A CORRESPONDENCE BETWEEN IRREGULAR FIELDS*

BY E. T. BELL

1. *Introduction.* Correspondences between fields are well known, and Dickson† has applied one to obtain a generalization of the theory of numbers. Here we give an instance of correspondence between irregular fields. An irregular field differs from a field only in the exclusion of an infinity of elements as divisors, instead of the uniquely excluded zero of a field. The postulates for an irregular field and numerous instances were given elsewhere.‡ The correspondence is established between the irregular field of all numerical functions and the irregular field of a certain infinity of power series with radius of convergence 1. For the series considered, addition and subtraction are interpreted as in the classical algebra of absolutely convergent series; multiplication and division receive wholly different interpretations. The simplest instance of the new multiplication is the process by which, when legitimate, a Lambert series is derived from a given power series.

It will be necessary for clearness to recall first a few definitions and theorems.

2. *The Irregular Field IF.* If $\xi(x)$ is uniform and defined for all integral values $n>0$ of x, $\xi(x)$ is called a *numerical function of* x. In what immediately follows, a relation involving n denotes the set of all relations obtainable from the given one by letting n range over all integers >0.

Let $\alpha(x), \beta(x), \cdots, \xi(x), \eta(x), \omega(x), \cdots$ be the set of all numerical functions of x, the *unit function* $\eta(x)$ and the *zero function* $\omega(x)$ being the unique functions defined by

$$\eta(1) = 1, \ \eta(x) = 0, \ x \neq 1, \tag{1}$$

$$\omega(n) = 0. \tag{2}$$

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† This Bulletin, vol. 23 (1916), p. 109.

‡ Annals of Mathematics, vol. 27 (1926), p. 511; *Algebraic Arithmetic* (Colloquium Publications of the American Mathematical Society, vol. 7, 1927).

By definition we assert that the two statements in each of the following pairs are formally equivalent (each implies the other); $\alpha(x)$, $\beta(x)$ are any numerical functions:

$$\alpha(n) = \beta(n), \quad \alpha = \beta; \tag{3}$$

$$\alpha(n) + \beta(n) = \xi(n), \quad \alpha + \beta = \xi; \tag{4}$$

$$a\alpha(n) = \xi(n), \quad a\alpha = \xi, \tag{5}$$

where a is a numerical constant. It is evident that $a\alpha(x)$ is a numerical function of x. Continuing, we define $\alpha\beta$ by

$$\sum \alpha(d)\beta(t) = \xi(n), \quad \alpha\beta = \xi, \tag{6}$$

the summation extending over all (d, t) where $d>0$, $t>0$ are integers such that $dt=n$.

The *elements* of the irregular field IF are $\alpha, \beta, \cdots, \xi, \eta, \omega, \cdots$. These elements are sufficiently defined by (3).

(7) DEFINITION. The element ξ of IF is *regular* or *irregular* according as $\xi(1) \neq 0$ or $\xi(1) = 0$.

(8) THEOREM. *If and only if ξ is regular, there exists a unique element ξ' of IF such that $\xi\xi' = \eta$, and ξ' is regular.*

The element ξ' in (8) is called the *reciprocal* of ξ (provided ξ is regular), and we write $\xi' = \eta/\xi$. The theorem was proved in a former paper,* where an explicit form of $\xi'(n)$ was given. In §3 we state an equivalent form, which is more convenient in certain applications of IF, including the present.

(9) THEOREM. *With equality as in* (3), *addition as in* (4), *scalar multiplication* (*and hence subtraction*) *as in* (5), *multiplication as in* (6), *division as in* (8), *the set of all $\alpha, \beta, \cdots, \eta, \omega, \xi, \cdots$ is an irregular field* (IF), *in which η is the unique unit element, ω the unique zero element, and the irregular elements are as in* (7).

3. *Explicit Form of Reciprocal.* Let ξ be an arbitrary regular element of IF. Write $\xi(1) = p$, $\xi\xi' = \eta$; so that $p \neq 0$ and ξ' is the reciprocal of ξ. We shall uniquely determine $\xi_0, \xi_1, \cdots$ in IF such that

* Tôhoku Mathematical Journal, May, 1920.

$$\xi' = \sum_{a=0}(-1)^a p^{-a-1}\xi_a, \tag{10}$$

where the upper limit of the summation may, if desired, be taken as ∞.

Let $m = p_1^{r_1} \cdots p_j^{r_j}$ be the resolution of $m > 1$ into powers of distinct primes $p_1, \cdots, p_j$; write $\pi(1) = 0$, $\pi(m) = r_1 + \cdots + r_j$. The following are definitions:

$$\xi_a(n) = 0, \qquad (a > \pi(n),\ n = 1, 2, \cdots); \tag{11}$$

$$\xi_0 = \xi^0 = \eta; \qquad \xi_b = 0, \qquad (b < 0). \tag{12}$$

Let $a_1, \cdots, a_s, P_1, \cdots, P_s$ be a particular set of integers such that, for $n > 1$ and $a > 0$ fixed,

$$\begin{gathered} n = P_1^{a_1}P_2^{a_2}\cdots P_s^{a_s}, \qquad a = a_1 + a_2 + \cdots + a_s, \\ 1 < P_1 < P_2 < \cdots < P_s, \quad 0 < a_1,\ 0 < a_2, \cdots, 0 < a_s. \end{gathered} \tag{13}$$

Then

$$\xi_a(n) = \frac{(\xi(P_1))^{a_1}(\xi(P_2))^{a_2}\cdots(\xi(P_s))^{a_s}}{a_1!a_2!\cdots a_s!}, \qquad (n > 1), \tag{14}$$

where the summation refers to all the distinct decompositions of n of the type defined in (13); the variables in the summation are $s, a_1, \cdots, a_s, P_1, \cdots, P_s$. The ξ_a in (10) are given by (11), (12), (13), and evidently $\xi'(n)$ is a finite sum of at most $\pi(n)$ terms. We may omit a detailed proof, as it follows readily from the paper cited in §2, end.

4. *I-Operations on Power Series*. Let α, β, γ be any elements of IF such that $\alpha\beta = \gamma$, and let each of the series

$$A(x) = \sum_{n=1}^{\infty}\alpha(n)x^n, \quad B(x) = \sum_{n=1}^{\infty}\beta(n)x^n$$

have radius of convergence 1. Then the series

$$C(x) = \sum_{n=1}^{\infty}\gamma(n)x^n$$

converges absolutely within the unit circle Γ.

For, if r is an integer > 0, and x_0 is in Γ, $|x_0^r| < 1$, and $\beta(r)\sum_{n=1}^{\infty}\alpha(n)x_0^{rn}$ converges absolutely. Hence

$$\sum_{r=1}^{\infty}\left[\beta(r)\sum_{n=1}^{\infty}\alpha(n)x_0^{rn}\right]$$

converges absolutely, and by (6) its sum is $C(x_0)$.

The *I-product* $A(x)$ $B(x)$ of $A(x)$, $B(x)$ is defined within Γ to be $C(x)$, and we write

(15) $$A(x)B(x) = C(x).$$

As a special case of (15), if $\beta = u$, where $u(n) = 1, (n = 1, 2, \cdots)$, the I-product $A(x)$ $B(x)$ is the Lambert series $\sum_{n=1}^{\infty}\alpha(n)x^n/(1-x^n)$.

Write

$$\sigma = \alpha + \beta, \quad S(x) = \sum_{n=1}^{\infty}\sigma(n)x^n.$$

Then $S(x)$ converges absolutely in Γ, and we define the *I-sum* of $A(x)$, $B(x)$ to be the ordinary sum,

(16) $$A(x) + B(x) = S(x).$$

From (14), (16) the I-difference is defined in an obvious manner. The following are definitions:

(17) $$U(x) = \sum_{n=1}^{\infty}\eta(n)x^n, \quad O(x) = \sum_{n=1}^{\infty}\omega(n)x^n;$$

$U(x)$ is called the *I-unit series*, and $O(x)$ the *I-zero series*; $U(x) = x$, $O(x) = 0$.

(18) THEOREM. *The set of all power series in x lacking the constant term and having radius of convergence* 1 *is a ring, say the I-ring, in which addition is as in* (16), *multiplication as in* (15), *and the unit, zero elements are* $U(x)$, $O(x)$ *respectively.*

The proof is immediate, on recalling the definition of a ring as a set closed under addition, multiplication, and subtraction. For example, if $A(x)$, $B(x)$, $C(x)$ are any elements of the I-ring, the distributive law asserts that

$$A(x)[B(x) + C(x)] = A(x)B(x) + A(x)C(x),$$

the indicated multiplications and additions being in the I-ring. But this is obvious from the definitions of the operations and (9). Generally, by the correspondence established in

(15), (16) between operations in the I-ring, and in IF, and the correspondence between IF and a ring of an abstract field, the theorem is proved.

The element $X(x)=\sum\xi(n)x^n$ of the I-ring is now defined to be *regular* or *irregular* according as $X'(x)=\sum\xi'(n)x^n$, where $\xi\xi'=\eta$, is or is not in the I-ring.

Let $A(x)=\sum\alpha(n)x^n$ be any element of the I-ring, and $B(x)=\sum\beta(n)x^n$ any regular element of the I-ring. Write $B'(x)=\sum\beta'(n)x^n$, where $\beta\beta'=\eta$. Then the *I-quotient* $A(x)/B(x)$ of $A(x)$ by $B(x)$ is defined by

$$A(x)/B(x) = A(x)B'(x); \tag{19}$$

$B'(x)$ is called the *I-reciprocal* of $B(x)$, and we write $B'(x) = U(x)/B(x)$. Combining (18), (19), we have the following theorem.

(20) THEOREM. *The set of all elements of the I-ring is an irregular field, say the I-field, in which division is as in* (19) *and the remaining fundamental operations as in* (18); *the irregular elements of the I-field are those of the I-ring.*

CALIFORNIA INSTITUTE OF TECHNOLOGY

ON TRI-RHAMPHOIDAL AND BI-OSCNODAL QUINTIC CURVES

BY HAROLD HILTON

In a recent paper,* T. R. Hollcroft says "For example, a quintic may have three rhamphoid cusps or two tacnode-cusps."

Now it is true that there is just one projectively distinct quintic with three rhamphoid cusps (or two, if we confine ourselves to real projections), namely

$$x:y:z = t^2(t - \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}) : t^2(t-1)^2(t - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}) : (t-1)^2(t + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}).$$

* This Bulletin, vol. 35 (1929), p. 847.