(15), (16) between operations in the I-ring, and in IF, and the correspondence between IF and a ring of an abstract field, the theorem is proved.

The element $X(x)=\sum\xi(n)x^n$ of the I-ring is now defined to be *regular* or *irregular* according as $X'(x)=\sum\xi'(n)x^n$, where $\xi\xi'=\eta$, is or is not in the I-ring.

Let $A(x)=\sum\alpha(n)x^n$ be any element of the I-ring, and $B(x)=\sum\beta(n)x^n$ any regular element of the I-ring. Write $B'(x)=\sum\beta'(n)x^n$, where $\beta\beta'=\eta$. Then the *I-quotient* $A(x)/B(x)$ of $A(x)$ by $B(x)$ is defined by

$$(19) \qquad A(x)/B(x) = A(x)B'(x);$$

$B'(x)$ is called the *I-reciprocal* of $B(x)$, and we write $B'(x) = U(x)/B(x)$. Combining (18), (19), we have the following theorem.

(20) THEOREM. *The set of all elements of the I-ring is an irregular field, say the I-field, in which division is as in* (19) *and the remaining fundamental operations as in* (18); *the irregular elements of the I-field are those of the I-ring.*

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ON TRI-RHAMPHOIDAL AND BI-OSCNODAL QUINTIC CURVES

BY HAROLD HILTON

In a recent paper,* T. R. Hollcroft says "For example, a quintic may have three rhamphoid cusps or two tacnode-cusps."

Now it is true that there is just one projectively distinct quintic with three rhamphoid cusps (or two, if we confine ourselves to real projections), namely

$$x:y:z = t^2(t - \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}) : t^2(t-1)^2(t - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}) \\ :(t-1)^2(t + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}).$$

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* This Bulletin, vol. 35 (1929), p. 847.

It is the quadratic transform of a three-cusped quartic with respect to a triangle, whose vertices lie on the quartic and each of whose sides passes through a cusp.*

But it is not true that a quintic can have two tacnode-cusps. For, taking ABC as triangle of reference and using suitable homogeneous coordinates, write down the general equation of a quintic with double points at A and B such that the tangents to both branches at these points are respectively AC and BC, while these tangents meet the curve four times at A and B. Then analyze the equation of the curve to obtain the conditions for the existence of two latent double points at each of A and B.† We see then that there are two types of quintic curve with two real oscnodes A, B, and tangents AC, BC.

One of them is

$$x(xy + az^2)^2 = y(xy + bz^2)^2,$$

or

$$x:y:z = t(t^2 + b)^2 : t(t^2 + a)^2 : (t^2 + a)(t^2 + b).$$

The analysis shows that for no value of a or b can one of the oscnodes become a tacnode-cusp.

The tangents at the oscnodes A, B meet at an inflexion C of the curve, the tangent being $a^2x = b^2y$. The remaining tangent from C to the curve, namely $x = y$, has its point of contact on AB.

The other type of quintic with oscnodes A, B and tangents AC, BC is

$$x(u + az^2)^2 + y(u + bz^2)^2 = 2z(u + az^2)(u + bz)^2,$$

where $u \equiv xy - z^2$; or

$$x:y:z = (1 - t)(b - t^2)^2 : (1 + t)(a - t^2)^2 : (a - t^2)(b - t^2).$$

The analysis shows that the oscnode at A becomes a tacnode-cusp if $a = 0$, and so for B. But we cannot have *two* tacnode-cusps, for the curve degenerates if both a and b are zero.

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* See Hilton's *Plane Algebraic Curves*, Clarendon Press, p. 120.

† Loc. cit., p. 134.