# INVARIANT POSTULATION*

BY T. R. HOLLCROFT

1. *Definition.* The postulation of a given manifold, simple or multiple, on a hypersurface is the number of conditions necessary in order that the hypersurface contain the given manifold. The manifold is uniquely determined both with respect to its nature and its position in the space involved. There exist among the coefficients of the hypersurface equations equal in number to the postulation. If the position of the given manifold is not defined, the parameters associated with its general position will occur in these equations. These parameters may be eliminated from the set of equations giving rise to a certain number of invariant relations involving the coefficients of the hypersurface. These invariants express the necessary and sufficient conditions that the hypersurface contain a manifold whose nature only is defined. The number of such invariants associated with a hypersurface and a manifold of given nature will be called the invariant postulation of that manifold on the hypersurface.

If $P$ is the postulation of a manifold $\phi$ on a hypersurface $f$ in $i$ dimensions, and $q$ is the number of independent conditions determining the position of $\phi$ in $i$ dimensions, the invariant postulation $I$ of $\phi$ on $f$ is given by the relation

$$I = P - q.$$

Algebraically, this means that $q$ arbitraries can be eliminated from $P$ equations in $P-q$ independent ways.

2. *Application.* The purpose of this paper is to show how certain geometric relations involving general hypersurfaces and manifolds are revealed by means of the concept defined above. Some of these relations have been found before by other methods, but most of them are new.

It is not claimed that evidence of the existence of certain relations given by this method is proof of such existence. This method, by virtue of the general expression for $I$, points out

* Presented to the Society, October 27, 1928.

geometric relations that might otherwise remain undiscovered. The formal proof of the existence of such relations must be established independently. This existence can be proved in many cases. In no case, general or special, however, has the author been able to establish the non-existence of a property suggested by this method.

Consider a hypersurface $f$ of order $n$ in $i$ dimensions, and a manifold $\phi$ of order $s$ and dimension $d$. The invariant postulation of $\phi$ on $f$ is determined by the values of $n$, $i$, $s$, $d$, when $s=1$ and in some cases for all values of $s$.

If $I>0$, a certain number $I$ of invariant relations must exist among the coefficients of $f$ in order that $f$ contain $\phi$, that is, in order to contain $\phi$, $f$ must be a special hypersurface. If $f$ is to remain a general hypersurface with no relations among its coefficients, then the invariant postulation of any manifold it may contain must be either zero or negative.

The numbers $n$, $i$, $s$, $d$, $P$, $q$ must all always be positive integers, but since $I=P-q$, it results that $I \leqq 0$ when $q \geqq P$ respectively.

CASE I. $I=0$.

When $I=P-q=0$, the postulation $P$ of $\phi$ equals the number of arbitraries $q$ defining the position of $\phi$. Algebraically, we have a set of $P$ equations containing $q$ arbitraries, $P=q$. Solving this system of equations, a finite number of sets of values of the $q$ arbitraries is obtained. Each of these sets of $q$ values defines the position of a manifold $\phi$ contained in $f$. Since, however, there are no conditions on the coefficients of $f$, $f$ remains an entirely general hypersurface. The manifolds $\phi$ are also general if so considered originally, but their positions in $i$-space are wholly defined by $f$, that is, as soon as a particular $f$ is chosen, the manifolds $\phi$ contained in it are thereby determined.

Then $I=0$ for a given $n$, $i$, $s$, $d$ indicates that a general hypersurface of order $n$ in $i$ dimensions contains a finite number of manifolds of order $s$ and dimension $d$. In general when $s>1$, characteristics of $\phi$ in addition to $s$ and $d$ enter into the determination of $I$.

CASE 2. $I<0$.

When $I$ is negative, it can no longer be interpreted as the number of invariant relations necessary for $f$ to contain $\phi$. In this case, therefore, $I$ is defined solely by the relation

$$I = P - q$$

and the geometric meaning of $I$, when negative, follows from this definition.

Let $q-P=a$. Since in this case, $q>P$, it results that $a>0$ and $I=-a$, that is, the number of arbitraries defining the position of $\phi$ is greater by $a$ than the number of algebraic equations in the system in which these arbitraries occur. There are, therefore, $\infty^a$ sets of values of the $q$ arbitraries that satisfy this system of equations and each set of values of the $q$ arbitraries defines the position of a manifold $\phi$ contained in the general hypersurface $f$.

Then $I+a=0, a>0$, for a given set of values of $n, i, s, d$, etc., indicates that a general hypersurface of order $n$ in $i$ dimensions contains $\infty^a$ manifolds of order $s$, dimension $d$, etc.

A familiar example of this case occurs for a simple point on a hypersurface in $i$ dimensions. The postulation of the point is unity and the number of arbitraries locating the point in $i$ dimensions is $i$. Then $P=1$, $q=i$, $I=1-i$, that is, a general hypersurface of any order in $i$ dimensions contains $\infty^{i-1}$ points.

3. *Examples*. The case of a simple point is given at the end of the preceding section. For a double point on a hypersurface, $I=1$. No hypersurface can have any singular point gratis.

The postulation of a line on a hypersurface of order $n$ in $i$ dimensions is $n+1$. The position of a line in $i$ dimensions is determined by the positions of any two points on it, that is, by $2(i-1)$ arbitraries. Then for a line on $f$

$$I = n - 2i + 3.$$

Setting $I=0$, it results that a general hypersurface of order $2i-3$ in $i$ dimensions contains a finite number of lines. Schubert* discovered this by other methods.

When $I+a=0$, $a>0$, the above value of $I$ gives

$$n = 2i - a - 3.$$

* H. Schubert, *Die n-dimensionale Verallgemeinerung der Anzahlen für die vielpunktig berührenden Tangenten einer punktallgemeinen Fläche m-ten Grades*, Mathematische Annalen, vol. 26 (1886), p. 73.

Then a general hypersurface of order $2i-a-3$ in $i$ dimensions contains $\infty^{a}$ lines.

For a double line on $F$, we have

$$I = i(n-2)+2.$$

When $I=0$, $n<2$ for all values of $i$, that is, no hypersurface can have a double line gratis.

For a set of $s\geqq 2$ coplanar lines on $f$, since the set contains $s(s-1)/2$ intersections,

$$P = s(2n-s+3)/2, \quad q = 2s+3(i-2),$$
$$I = \tfrac{1}{2}s(2n-s-1)-3(i-2).$$

When $I=0$, there results

$$n = \frac{1}{2}(s+1)+\frac{3(i-2)}{s}\cdot$$

This is the order of a general hypersurface in $i$ dimensions that contains a finite number of sets of $s$ coplanar lines. This holds for all values of $i$ and $s$ such that $2\leqq s\leqq n$ and such that $n$ is a positive integer.

Substituting $s=3$ in the above value of $n$, there results $n=i$. Then a general hypersurface of order $n$ in $n$ dimensions contains a finite number of sets of three coplanar lines. The cubic surface in ordinary space is a special case of this.

The maximum number of coplanar lines that may occur on a hypersurface of order $n$ is $n$. Setting $s=n$ in the above value of $n$ and solving for $n$ we obtain

$$n = \tfrac{1}{2}[1+(24i-47)^{1/2}].$$

For a value of $i$ such that this formula gives a positive integral value of $n$, a general hypersurface of order $n$ in $i$ dimensions contains a finite number of sets of $n$ coplanar lines. For $i\leqq 24$, the values $(i, n)$ satisfying these conditions are (3, 3); (4, 4); (7, 6); (9, 7); (14, 9); (17, 10); (24, 12).

In case 2, $I+a=0$, $a>0$, there results

$$n = \frac{1}{2}(s+1)+\frac{3(i-2)-a}{s}\cdot$$

This defines the order of a general hypersurface in $i$ dimensions that contains $\infty^a$ sets of $s$ coplanar lines. The formula is satisfied by $n=s=a=2$ and $i=3$, that is, a quadric in 3-space contains $\infty^2$ sets of two coplanar lines as has long been known. No distinction is made here or elsewhere between real and imaginary lines. Setting $a=s=3$, we obtain $n=i-1$, that is, a general hypersurface of order $i-1$ in $i$ dimensions contains $\infty^3$ sets of three coplanar lines.

The postulation of a general plane curve of order $s \geqq 2$ on $f$ is the same as that of $s$ coplanar lines. For a plane curve of order $s$ in $i$-space, $q=s(s+3)/2+3(i-2)$ since the equation of a plane curve contains $s(s+3)/2$ arbitraries and the equations of a plane in $i$-space contain $3(i-2)$ arbitraries. Then for a general plane curve of order $s$ on $f$

$$I = s(n - s) - 3(i - 2).$$

The value of $I$ for a plane curve shows that the invariant postulation of a plane curve of given order on a surface of given order is equal to the invariant postulation of its residual on that surface. This is true because the existence of one is sufficient for the existence of the other. The postulations of the two are, however, different because different numbers of conditions are necessary to determine them.

The above expression for $I$ does not hold for $s=1$ and therefore it does not hold for $s=n-1$, since the residual of a plane curve of order $n-1$ is a line. If a hypersurface contains $\alpha$ lines, $\alpha$ finite or infinite, it also contains $\alpha \cdot \infty^{i-2}$ plane curves of order $n-1$ since $\infty^{i-2}$ planes pass through each line.

When $I=0$, $n=s+3(i-2)/s$. For $s=2$, $n=3i/2-1$. Then for every even value of $i$, a general hypersurface of order $3i/2-1$ contains a finite number of conics. For $s=3$, $n=i+1$, that is, a general hypersurface of order $i+1$ in $i$ dimensions contains a finite number of plane cubics.

When $I+a=0$, $n=s+(3i-a-6)/s$. If $i$ and $a$ are both odd or both even, hypersurfaces of order $n=(3i-a)/2-1$ contain $\infty^a$ conics. Also if $a$ is a multiple of 3, hypersurfaces of order $n=i+1-a/3$ contain $\infty^a$ plane cubics. If $s=n$, $3(i-2)=a$, that is, $f$ contains as many plane sections of order $n$ as there are planes in $i$-space.

The postulation of a linear manifold of dimension $d$ in $i$ dimensions is $(n+1)(n+2) \cdots (n+d)/d!$. The position of this manifold in $i$ dimensions is determined by $(i-d)(d+1)$ conditions. Then for such a manifold on $f$

$$I = (n+1)(n+2)\cdots(n+d)/d! - (i-d)(d+1).$$

For $d=2$, this becomes the invariant postulation of a plane on $f$

$$I = (n+1)(n+2)/2 - 3(i-2).$$

In this case, when $I=0$, on solving for $n$ we obtain

$$n = \tfrac{1}{2}\left[(24i-47)^{1/2} - 3\right].$$

This expression is less by two than the expression for the order of a hypersurface containing $n$ coplanar lines. Then in each dimension in which occurs a general hypersurface of order $n$ a finite number of whose plane sections can be composed entirely of lines, there is also a general hypersurface of order $n-2$ that contains a finite number of planes.

When, in the case of a plane, $I=0$, if $n$ is of the form $3c+1$ or $3c+2$, $i$ is always a positive integer. This implies that for all values of $n$ not divisible by three, general hypersurfaces of order $n$ in $i=(n^2+3n+14)/6$ dimensions contain a finite number of planes.

The above methods may be applied to manifolds of any order in any number of dimensions. The foregoing examples are some of the simplest applications. As stated previously, the fact that $I\leqq 0$ does not prove the existence of manifolds on general hypersurfaces. To investigate such geometric properties of hypersurfaces of all orders in all dimensions separately would be practically impossible. This method aids in selecting certain cases that merit further examination.

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