COBLE ON ALGEBRAIC GEOMETRY AND THETA FUNCTIONS

Algebraic Geometry and Theta Functions. By Arthur B. Coble. American Mathematical Society, Colloquium Publications, vol. X, New York, 1929. vii+ 282 pp.

The well known classical treatise by Krazer on the theory of θ functions contains several beautiful chapters dealing with the applications of this theory to algebraic geometry (in the largest sense of the word), but on the whole this treatise is more analytic than geometric in character. In it, page after page, swarms of complicated formulas and relations follow each other, rarely illuminated by a geometric interpretation. One can say, without fear of exaggeration, that all the geometric applications of this theory to algebraic curves and varieties made since Riemann and Weierstrass (Hurwitz, Poincaré, Schottky, Wirtinger, etc.) are absent in Krazer's treatise, or at most are only mentioned in short historical notes.

Having pointed out this gap in Krazer's treatment, we have thus indicated in advance the principal merit of Coble's new volume: the geometric spirit, which seeks to discover the geometric reality hidden behind the formal properties of the functions θ. The picture of this geometric reality, which the author draws before our eyes, is as varied as it is colorful: applications to the geometry on algebraic curves and surfaces; connections, as elegant as unexpected, with the Cremona transformations of the plane and of the space; analogies with the finite modular geometry and group theory aspect of the different geometric problems. It is true that the author, an expert algebraist true to the best traditions of the symbolic calculus of Clebsch-Aronhold, utilizes here and there a formidable algebraic apparatus, where the geometer would have preferred a simplified synthetic treatment. But this method has the positive advantage of facilitating the reading for those who see geometry best through algebraic glasses.

The main subject of this volume is the study of the canonical sextic C^6 of genus 4 in the space S_3, and in particular the determination of its tritangent planes. The author makes use of Wirtinger's representation of the C^6 as the locus W of the diagonal points of the quadruples of points of a g_4^1 on a general quartic curve. The locus W, the noted sextic of Wirtinger-Caporali, has 6 nodes at the vertices of a complete quadrangle. If W is given, then one of the $2^8-1=255$ systems of the contact quadrics of C^6 and all of the 28 tritangent planes contained in the system are rationally known. By means of the linear system of the ∞^3 adjoint cubics of W the plane is mapped upon one of the 255 Cayley cubic surfaces on C^6.

The consideration of Wirtinger's sextic does not lead, however, to the determination of all the 120 tritangent planes of C^6. It is here that Coble's analysis connects with the investigations by Schottky, relative to the modular functions of genus $p=3,4$. Schottky has shown that the coordinates of the 8 base points of a net of quadrics in S_3 can be expressed in terms of the zero values of the functions θ of genus 3 (modular function of genus 3). Similarly, taking as starting point the θ-relations in the case $p=4$, Schottky defines a special set of 10 points in S_3, the geometric peculiarity of which lies in the fact that these points can be taken as the 10 nodes of a surface of the 4th order, known as Cayley's symmetroid. The author investigates very closely the two sets, denoted by P_8^3 and P_{10}^3 respectively, and shows that they enjoy, in their respective fields, very similar properties both with respect to the transformations of the moduli and to Cremona transformations. In the case $p=3$ the set P_8^3 gives the complete solution of the problem of the determination of the 28 double tangents of the general plane quartic of genus 3. Shall we expect the set P_{10}^3 to play a similar part in the case $p=4$, as far as the problem of the tritangent planes of the C^6 is concerned? With this question, left unanswered, the author's investigation of that particular problem terminates, although the author declares himself strongly in favor of the belief that a suitable geometric construction in S_3, which should associate with the set P_{10}^3 a curve of genus 4, must not fail to bring forth the connection between Cayley's symmetroid and the problem of the tritangent planes of C^6.

Without discussing the conjecture of the author, which seems to us very ingenious indeed, we want to make the following remark. In the case $p=3$ the determination of the double tangents depends on a geometric construction in the space S_3, and not in the space S_2 of the relative canonical curve. Would it not be as logical to expect that in the case $p=4$, the space S_4 and not S_3 should be the natural site of the desired geometric configuration? Apart from conjectures, the problem formulated by the author is a most interesting one, especially in view of its possible generalization to any value of the genus p.

The author treats very completely the case of a special C^6, characterized by the condition of lying on a quadric cone, or, what is the same, by the condition that the two g_3^1's, which on a general C^6 are distinct and are each the residual of the canonical series with respect to the other, coincide into one self-residual g_3^1. The equivalent transcendental condition is $\theta\ ((0))=0$ (Weber). This C^6 is birationally equivalent to the locus of the invariant points of a Bertini involution, which locus is a curve of order 9 with 8 triple points at the fundamental points of the involution. The 120 degenerate sextics with nodes at the fundamental points give all the proper solutions of the problem. Schottky has given a direct definition of the set P_8^2 in terms of the modular functions. Coble's investigation completes this formal definition from a geometric point of view.

The questions outlined above, although of evident geometric interest, are of a special character. But in the treatment of these questions the author finds an opportunity to establish very general and important connections between Cremona transformations on one side and the theory of θ-functions on the other. This parallelism between the two fields constitutes perhaps the most original part of the author's work. There is first and above all the concept of *congruent* sets of points in the plane or in space. Two ordered sets of m points each, P_m and Q_m, are said to be congruent if there exists a Cremona transformation T with $\rho \leqq m$ F-points, such that the F-points of T are in P_m, the F-points of T^{-1} are in Q_m, and the $m-\rho$ remaining points of the two sets are corresponding points in T. The number of projectively distinct sets congruent to a given set P_m is in general infinite, but becomes finite for projectively particular sets and also, in the plane, when $m<9$. The study of some of the most important examples of these particular sets P_m constitutes the foundation upon which the most essential part of the investigation is based, and it is here that the correlation between Schottky's investigations and the author's own results is most expressive. This is due to the fact that, according to Schottky, the coordinates of the points of these sets can be expressed in terms of the modular functions. Now, Coble establishes the beautiful theorem that the sets of points thus defined are carried by period transformations into congruent sets of points. This theorem is in correlation with the following general principle:

Let $g_{m,k}$ be the group of linear integral transformations, associated with the different types of Cremona transformations of S_k with $\rho \leqq m$ F-points (Kantor). If the coefficients of the transformations of $g_{m,k}$ are reduced mod 2, a finite group is obtained which is isomorphic to the group of the half-period transformations for a suitable value of p, or to a subgroup of this group.

The author develops a very elegant and important representation of the group $g_{m,k}$ as an infinite (discontinuous) group $G_{m,k}$ of Cremona transformations in a space $S_{m(m-k-2)}$, by mapping projectively distinct sets P_m of S_k upon the points of the said space. Each transformation of $G_{m,k}$ represents a type of Cremona transformation of S_k with $\rho \leqq m$ F-points, and corresponding points of the transformation in $S_{k(m-k-2)}$ arise from congruent sets in S_k. The group $G_{m,k}$ describes, so to speak, the purely discontinuous parameters on which the Cremona transformations of S_k depend.

Another excellent aspect of the volume is the use of the finite modular geometry in the exposition of the theory of characteristics. The representation of the half-period characteristics and of the θ-characteristics by means of points and quadrics of a modular space S_{2p-1} has a great descriptive and didactic value. The elements of the theory of characteristics, which form the content of the second part of Krazer's treatise, are presented in a very simple and suggestive form, thanks to the use of the language of finite geometry.

Given the subject and the nature of the problems treated, the book is not exactly, and one could not expect it to be, easy reading. Perhaps the abundance of details does not make the reading easier. However, the general lines of the exposition are well planned and the style is clear and concise. We feel that, thanks to its rich geometric content and originality of treatment, Coble's work is a really important contribution to the theory and application of the θ-functions, and will be well accepted by competent readers.

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