DICKSON'S THEORY OF NUMBERS

Introduction to the Theory of Numbers. By Leonard Eugene Dickson. The University of Chicago Press, 1929. vi+183 pp.

The appearance in English of a thoroughly sound and scholarly introduction to the theory of numbers, illuminated by many striking flashes of originality in both substance and presentation, is no commonplace in the progress of mathematics. For while it is true, as the author says, that "during twenty centuries the theory of numbers has been a favorite subject of research by leading mathematicians and thousands of amateurs," the theory has had fewer books, good, bad or mediocre, written around it than any other of the recognized major fields of mathematics. In English there are probably less than six treatises devoted entirely to the subject, and these range from the uncompleted work of Mathews to brief descriptions of the rudiments of the theory, quite inadequate to give a reader a working knowledge of even the meagre field covered. Although the merits of some of these older books have long been recognized, they belong to a past age, and are in fact hopelessly oldfashioned. The like holds also in a lesser degree for some of the great European treatises. New works will not destroy the inherent values of the old, but the need for something livelier than a lukewarm rehash of the imperishable *Disquisitiones Arithmeticae* of Gauss as an introduction to the theory of numbers is badly needed, and this Professor Dickson has supplied.

With a uniquely comprehensive and detailed grasp of the history of the subject, the author is in a position to select from the bewildering mass of material available those parts of highest interest in which more than a mere semblance of vitality still resides. So far as is possible with the means to which he necessarily has restricted himself in an introduction, namely nothing more advanced than college algebra as understood in America, the author has presented a living picture of the classic theory as left by Gauss and of more than one outstanding advance since Gauss wrote his masterpiece 130 years ago. For example, Thue's great theorem and its generalizations (1909–1921) are here presented in English for the first time, and with extremely simple proofs. We shall recur to this later.

Again, as a long overdue deviation from the Gaussian tradition, Dickson has used his unrivalled historical knowledge to restore to their proper position certain fruitful concepts which have been neglected for over a century. Of these may be noted in particular the idoneal numbers of Euler, the return to general rational integer coefficients in binary quadratic forms as advocated by Kronecker, and the restoration of pre-Gaussian diophantine analysis, the origin of the whole vast theory of forms, to a place of prominence. With this restoration of the old in modern shape, arithmeticians will have but slight excuse for declaring that certain problems are no longer of importance merely because all methods, ancient and modern, have thus far failed to exhibit solutions.

Some idea of the scope of this work can be gained from the following paraphrase of its contents: Chapter I, Divisibility; II, Theory of Congruences;

III, Quadratic Reciprocity; IV, Introduction to Diophantine Equations; V, Binary Quadratic Forms; VI, Certain Diophantine Equations; VII, Indefinite Binary Quadratic Forms; VIII, Solution of $ax^2+by^2+cz^2=0$ in Integers; IX, Composition and Genera of Binary Quadratic Forms; X, The Thue-Siegel Theorems; XI, Minima of Real, Indefinite Binary Quadratic Forms.

Many matters in these concise, beautifully clear chapters would call for more extended comment in a complete notice; here only a few can be selected to give the flavor of the whole to different tastes.

Progress is rapid. By page 5, for instance, a reader will have sufficiently grasped the elementary properties of primes to be able to prove readily that there are infinitely many primes $4n-1$, as stated in example 12. On page 9, example 11, the converse of Fermat's theorem is stated with a hint of the proof. By page 14, the number of roots of a congruence has been disposed of, and by page 20, the essential theory of primitive roots is out of the way. On page 21 new ground is broken in the theory of residual polynomials and congruences. Here Dickson presents, among other things, an elegant reworking of the inclusive theory of Kempner (1921) in about five pages. The exercises on page 28 contain several general theorems of interest. In Chapter III, 8 pages (small ones at that), quadratic residues, including the reciprocity laws for the symbols of Legendre and Jacobi, are treated with all sufficient detail. The third proof of Gauss is given, also Eisenstein's geometrical equivalent of the second part.

In Chapter IV there is a refreshing departure from the tradition of most writers of treatises on the theory of numbers: diophantine analysis is introduced in a simple but severely scientific dress. Most of the general theorems in this chapter are from the author's original work of the past few years, although this is not stated in the text. The good old times of diophantine analysis, when writers were content with *some* solutions of their problems, are gone, let us hope, forever. Just as no decent analyst today is content with anything but the best conditions under which his theorems hold, so the devotee of the theory of numbers demands *all* solutions or *none*. This is a hard ideal to attain, and exceptional circumstances may justify the publication of partial results. Dickson however has set himself the whole problem or nothing, and he solves it for such things as *all* integral solutions (x, y, z) of $ax^2+bxy+cy^2=ez^2$ where a, b, c, e are integers, $e\neq 0$, b^2-4ac not square, in terms of any given solution (ξ, η, ζ). As an exercise on page 49, the reader is asked to apply the general principles developed to all rational solutions of $aX^2+bY^2+cZ^2=e$, given one solution, and to observe the radical difficulty in attempting to pass thence to all integral solutions. Included in this chapter, for the first time apparently in a treatise, are the curious and interesting theorems on sums of integers having equal sums of like powers. Among other gems is an extremely simple derivation of all rational solutions of $W^3+X^3+Y^3+Z^3=0$. It is pointed out that the complete solution of $X^4+Y^4=Z^4+W^4$ in integers is yet to be found.

In Chapter V, binary quadratic forms $ax^2+bxy+cy^2$ are fully treated with respect to equivalence, reduction, number of representations, and genera, all in 28 pages. Following custom, the theory is carried first as far as possible until the separation into definite and indefinite forms becomes inevitable. As in the rest of the book, this subject, although long classical, is presented with remarkable freshness and, a gift beyond all price to successive generations of students who will not henceforth be compelled to hack their blind path through scores of pages of unnecessary details, astonishing brevity and equally astonishing clarity. To mention only two novel details, which will be of great service to those working on numbers of representations in n-ary quadratic forms ($n>2$), we may cite the table on page 85 of reduced, positive, primitive forms of discriminant $-\Delta$ with a single class in each genus, for $\Delta<400$, and the statement on page 89 of those Δ, where $400<\Delta<23000$, for which there is a single class of positive, primitive forms of odd discriminant $-\Delta$. In the same direction, the results of many of the exercises in this chapter are of great value as savers of labor.

In Chapter VI, diophantine equations again appear, this time first in a thoroughgoing investigation of all integral solutions of $x^2-my^2=zw$. The discussion is a model of what modern diophantine analysis should be and may become under competent hands. To discuss $ax^2+bxy+cy^2=zw$, a slight levy is made on the resources of the preceding chapter. The method of Euler and Lagrange (forms that repeat under multiplication) is glanced at and discarded on account of its lack of generality. In this connection, on the algebraic side, the author might well have cited his own fundamental papers. Arithmetical generality, it would seem, can be attained in some instances at least by the machinery of ideals. But, even in these instances, it remains to be shown that the problem is actually solved in a constructible manner and not merely disguised by a restatement in sophisticated and artificially simpler language.

Indefinite binary quadratic forms receive in Chapter VII (18 pages) an adequate treatment equally as brief and as clear as that of positive forms in V. The reviewer does not recall another presentation of this classical theory, which in spite of its general beauty is rather messily detailed in spots, approaching this one in economy of space and effort.

The famous equation $ax^2+by^2+cz^2=0$ of Legendre and Gauss is the subject of Chapter VIII. Dickson essentially follows Dedekind's proof of the criteria for non-trivial solutions. The proof requires only four pages. Modern work by Ramanujan and Dickson and his students on universal forms (those representing all integers) and regular forms (universals except as to integers in certain arithmetic progressions), has demanded a closer scrutiny of certain aspects of ternary quadratic forms than has been customary in the literature. Together with the classical theory, the ground for these newer ideas is carefully prepared in this chapter.

Composition and the related problems of genera, which can be made quite tedious if clumsily handled, are elegantly treated in Chapter IX. What will probably strike a practised eye at once in this presentation is the complete absence of long formulas. Dedekind's proof of Gauss' theorem on duplication is given in less than two pages, both clear.

A new topic, whose difficulties have taxed experts in the subject, is discussed in the five pages of Chapter XI. Here it is a question of finding the lower bound of the absolute values of the numbers represented by any real indefinite binary quadratic form of given discriminant. The locus classicus here is Markoff's paper of 1879, which extends the main result to an infinitude of forms each having a minimum exceeding a certain prescribed limit. Dickson produces short proofs of the key theorems, which are simpler than any others in the literature. An exposition of Markoff's extension is promised in the author's forthcoming sequel to this *Introduction*. The whole subject of minima has taken on a new importance in the light of modern methods.

To return to Chapters IX, X, in some respects the crown of the work. Outside of the brilliant achievement in the modern analytic theory of numbers at the hands of Hardy, Landau, Littlewood and their followers, arithmetic since Gauss has few results to offer that are so completely general and altogether satisfying as the theorem (1909) of Axel Thue. It is not a theory, it is a theorem. Some may prefer the abstract beauty of the theory of ideals, overlooking the fact that so far it has failed to solve the problem for which it was first created. The appeal of Thue's theorem is of a totally different order: a general problem had been outstanding for centuries; Thue devised a weapon of extraordinary power for attacking special cases of the problem and, in a significant sense, gave a solution of the general problem. Its entire simplicity marks Thue's theorem as one of the universal, natural things which are the very essence of the theory of numbers.

Here it is, as stated in Dickson's *Introduction*.

THEOREM 107. Let $f(z)=a_nz^n+\cdots+a_0$ be an irreducible polynomial of degree $n\geqq 3$ with integral coefficients. Consider the corresponding homogeneous polynomial $H(x, y)=a_nx^n+a_{n-1}x^{n-1}y+\cdots+a_1xy^{n-1}+a_0y^n$. If c is an integer, $H(x, y)=c$ has either no solution or only a finite number of solutions in integers.

THEOREM 116. Theorem 107 holds also if we omit the assumption that $f(z)=0$ is irreducible, but assume that all its roots are distinct, and $c\neq 0$.

Siegel's generalization (Theorem 113) provides a generalization of 107 different from 116, concerning $H(x, y)=G(x, y)$, where $G(x, y)$ is another polynomial (not perfectly general).

Thue's own proof of his great theorem is elementary and hard (a lacuan was first supplied by Maillet). Landau's presentation (following largely Siegel) in his classic *Vorlesungen* appeals to algebraic numbers and Farey fractionss and it demands considerable sophistication on the part of the reader. Dickson', signal service has been the reduction of the proof to a strictly elementary form. The word "elementary" is here meant in the sense of college algebra.

In closing this notice of an outstanding contribution to modern mathematical literature, we may emphasize again its clearness, its width of vision, and the unerring precision with which it consistently "hews to the line." The problems have been mentioned more than once. They are approachable and, in the opinion of the reviewer, afford the one decent set of exercises in existence on the, theory of numbers. The first volume of a sequel to the present *Introduction* containing much original work of the author and his students, is already in press. It is safe to predict that readers will find in it a worthy successor to the present. Dickson's work is above all else modern, independent in outlook, and thoroughly alive. Perhaps he will give us a book on the theory of algebraic numbers after the present series is completed. For the analytic theory we have Landau, but nothing comparable to it in English.

Siegel, I believe (if I am mistaken, I apologize), has said that the next great step in mathematical progress will be the burning of all books on mathematics. Should that somewhat spectacular step be taken, let us hope that Dickson's *Introduction* and a few other works of mathematical art escape.

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THE NEW MECHANICS

Einführung in die Wellenmechanik. By Louis de Broglie. Translated into German by Rudolf Peierls. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1929. iv+221 pp.

La Nouvelle Mécanique des Quanta. By George Birtwistle. Translated into French (and augmented by four appendices) by M. Ponte and Y. Rocard, with a preface by J. Hadamard. Paris, Albert Blanchard, 1929. vi+333 pp.

Gruppentheorie und Quantenmechanik. By Hermann Weyl. Leipzig, S. Hirzel, 1928. viii+288 pp.

About two years ago the new wave mechanics acquired remarkable experimental support through the discovery of the hitherto quite unknown phenomenon of diffraction of electronic streams by crystals. This discovery affords a genuine counterpart to the earlier discovery of the photoelectric effect, for the first shows that heretofore one aspect of the nature of matter has been left out of consideration while from the second it appears that one aspect of the nature of light had long been overlooked. It is now generally recognized that the behavior of light is corpuscular as well as undulatory; the diffraction of an electronic stream compels us to recognize that the behavior of matter is undulatory as well as corpuscular.

The older quantum theory failed to meet the dilemma of this two-fold character of light and of course did not consider at all the corresponding duality in the nature of matter; and it found no explanation of the presence of half quantum numbers in the formulas of the Zeeman effect and band spectra. A new theory was therefore inevitable. The first step was taken by Louis de Broglie in his dissertation (1924). The wave mechanics initiated by de Broglie received remarkable development at the hands of E. Schrödinger. Another line of development is due to W. Heisenberg.

De Broglie's *Wellenmechanik* furnishes the best introduction to the new quantum mechanics which has come to the reviewer's attention. A reader who desires to begin with a more elementary exposition would probably do well to use the second edition of *Materiewellen und Quantenmechanik* by A. Haas (1929); in this book the simpler aspects of the new theories are presented in an illuminating way. The exposition by de Broglie would then serve to complete an introduction to these most remarkable developments of the newer physics.

The starting point of de Broglie's wave mechanics has its origin in his purpose to develop the theory in such a way that there shall be an intimate connection between the conception of a corpuscle and that of periodicity in