ON A COMPLETE CHARACTERIZATION OF THE SET OF POINTS OF UNBOUNDED GRADE OF AN ARBITRARY SURFACE*

BY HENRY BLUMBERG

Let $z=f(x, y)$ be an arbitrary surface S, in the sense that $f(x, y)$ is an arbitrary one-valued real function of the real variables x and y. By the *grade* of a segment joining two points A and B, we understand the absolute value of the tangent of the angle which AB makes with the xy-plane. The point $A=(\xi, \eta, \zeta)$ of the surface S is said to be of *bounded grade*—or S is said to be of bounded grade at the point (ξ, η)—if the grade of AB is bounded for all $B=(x, y, z)$ of S at a sufficiently small "horizontal" distance $[(x-\xi)^2+(y-\eta)^2]^{1/2}$ from A. If A does not satisfy this condition, S is said to be of unbounded grade at (ξ, η). It is the object of the present paper to prove the following theorem, which identifies the aggregate—for the totality of arbitrary surfaces—of sets of points of unbounded grade with the aggregate of sets of type G_δ.†

THEOREM.‡ *The set of points (x, y) at which an arbitrary surface $z=f(x, y)$ is of unbounded grade is a G_δ. Conversely if a G_δ is given, there exists a surface $z=f(x, y)$ such that this G_δ is identical with the set of points (x, y) where the surface is of unbounded grade.*

PROOF. If $P=(x, y)$ is a point at which the given surface S, represented by $z=f(x, y)$, is of unbounded grade, we may, for every positive integer n, find a point $P_n=(x_n, y_n)$ such that the distance $d(PP_n)$ between P and P_n is less than $1/n$, and $g(f, PP_n)>n$, understanding by $g(f, PP_n)$ the grade of the segment joining the points of S corresponding to P and P_n. Enclose P and P_n in a circle $C_P^{(n)}$, regarded as made up only of

* Presented to the Society, April 16, 1927.

† A G_δ is the product of $\aleph_0$ open sets. The notation is due to Hausdorff.

‡ The direct part of this theorem is stated by W. H. Young for the case of a function of one variable; see Arkiv för Matematik, Astronomi och Fysik, vol. 1 (1903).

interior points, of diameter $<2d(PP_n)$. Let G_n equal the sum of the $C_P^{(n)}$ for n fixed and P ranging over the set U of points at which S is of unbounded grade; and let $T=\prod_1^\infty G_n$. We show that $T\equiv U$. Of course, since every point of U is in G_n for every n, we have $U\subseteqq T$. Suppose Q is in $C_P^{(n)}$. Then either $|f(Q)-f(P)|$ or $|f(Q)-f(P_n)|$, where $f(P)=f(x, y)$, with similar meaning for $f(Q)$ and $f(P_n)$, is not less than $\frac{1}{2}|f(P)-f(P_n)|$. Since $d(QP)$ and $d(QP_n)$ are both less than $2d(PP_n)$, we conclude that either $g(f, QP)$ or $g(f, QP_n)>g(f, PP_n)/4>n/4$. It follows that if Q is in some $C_P^{(n)}$ for every n, then S is of unbounded grade at Q. Hence $T\subseteqq U$, and therefore $T\equiv U$.

To prove the converse part of the theorem, we suppose that $\prod_1^\infty G_n$ is a given product of open sets G_n lying in the xy-plane. Let $G^{(n)}=\prod_1^n G_n$, and $F_n=$ the complement of $G^{(n)}$ with respect to the xy-plane. In terms of these F_n, we shall define the surfaces $z=f_n(x, y)$; and $z=f(x, y)=\sum_1^\infty f_n(x, y)$ will be the required surface having bounded grade at the points of $\sum_1^\infty F_n$ and unbounded grade at the points of $\prod_1^\infty G_n$.

To this end, we suppose that $T_n=\{Q^{(n)}\}$ is, for every positive integer n, a system of non-overlapping squares $Q^{(n)}$ lying in $G^{(n)}$, such that every point of $G^{(n)}$ is in the interior or on the boundary of at least one $Q^{(n)}$ of T_n. We define $f_n(x, y)$ as 0 at the points of F_n and at the boundary points of the $Q^{(n)}$; if $P=(x, y)$ is an interior point of the square $Q^{(n)}$, we set

$$f_n(P)=f_n(x, y)=\rho_n d_P^{(n)},$$

where $d_P^{(n)}$ is the distance from P to the boundary of $Q^{(n)}$, and ρ_n is a number, depending on n but not on the varying $Q^{(n)}$ of T_n, and subject to certain relations to be stated presently. We suppose furthermore that T_{n+1} is a "subdivision" of $T_n\{n=1, 2, \cdots\}$ in the sense that every $Q^{(n+1)}$ of T_{n+1} lies in just one $Q^{(n)}$ of T_n. Let $2q^{(n)}$, which may vary, as $Q^{(n)}$, n fixed, varies, be the length of side of $Q^{(n)}$. Then, as we may, we select the ρ_n and $q^{(n)}$ in such a way that

(a) $$\rho_n/\sqrt{2} > M_{n-1} + n(n \geqq 2); \ \rho_1 = 1;$$

(b) $$\rho_n q^{(n)} < d_n^2/2^n;$$

(c) $$\rho_n q^{(n)} < q^{(n-1)}.$$

Here M_n represents the upper boundary, which is evidently finite, of the grade of a segment with end points on the surface

$$z = s_n(x, y) = \sum_1^n f_\nu(x, y),$$

for example, $M_1 = 1$; and d_n, which depends, for n fixed, on $Q^{(n)}$, is the minimum distance from the boundary points of $Q^{(n)}$ to F_n; moreover, inequality (c) is to be understood as demanded only in case the square $Q^{(n)}$ of side $2q^{(n)}$ lies in, or has at least one boundary point in common with the square $Q^{(n-1)}$ of side $2q^{(n-1)}$. We now set $f(x, y) = \sum_1^\infty f_n(x, y)$, and observe that f exists. For, by the definition of f_n and inequality (b), if P is an interior point of some $Q^{(n)}$ of T_n, then

$$f_n(P) = \rho_n d_P^{(n)} \leqq \rho_n q^{(n)} < d_{nP}^2/2^n,$$

where d_{nP} is the minimum distance from P to F_n. Since d_{nP} does not increase with n, and $f_n(P) = 0$ if P is interior to no $Q^{(n)}$ of T_n, it follows that $\sum f^n(P)$ is convergent. We shall now prove that f has the required properties.

First, let P be a point of F_n. If $\nu > n$, and P' is a point of G_ν lying in the interior of the square $Q^{(\nu)}$ of T_ν, we have, by inequality (b),

$$g(f_\nu, PP') < \frac{\rho_\nu q^{(\nu)}}{d(PP')} < \frac{\rho_\nu q^{(\nu)}}{d_\nu} < \frac{d_\nu}{2^\nu} < \frac{d(PP')}{2^\nu}\,.$$

If $P' \neq P$ is interior to no $Q^{(\nu)}$ of T_ν, $f_\nu(P') = 0$, and since $f_\nu(P) = 0$, we have $g(f_\nu, PP') = 0$. It follows, if $r_n(x, y) = \sum_{n+1}^\infty f_\nu(x, y)$, that $g(r_n, PP') < d(PP')$ for every point $P' \neq P$. Since the surface $z = s_n(x, y)$ is of bounded grade at every point, and

$$f(x, y) = s_n(x, y) + r_n(x, y),$$

we conclude that the surface $z = f(x, y)$ is of bounded grade at every point of every F_n, and therefore of bounded grade at every point of $\sum F_n$.

Now let P be a point of $\prod G_n$, and $Q^{(n)}$ a square of T_n containing P in its interior or on its boundary. Then there is a point P' in $Q^{(n)}$ such that $d(PP') \geqq q^{(n)}/2$ and $g(f_n, PP') \geqq \rho_n/\sqrt{2}$. Since $g(s_{n-1}, PP') \leqq M_{n-1}$, we have, by inequality (a), the relation $g(s_n, PP') > n (n \geqq 2)$. Let ν be an integer greater than n. If P and P' both belong to $G^{(\nu)}$ there are two squares of T_ν, possibly identical, the one containing P and the other P'; let

$Q^{(\nu)}$ be the one with the larger (or at least not the smaller) side $q^{(\nu)}$. Then

$$g(f_\nu, PP') \leqq \rho_\nu q^{(\nu)}/d(PP') < 2\rho_\nu q^{(\nu)}/q^{(n)}.$$

Moreover $Q^{(\nu)}$ lies in a $Q^{(\nu-1)}$ and this in turn in a $Q^{(\nu-2)}$ and so on down to $Q^{(n+1)}$, which lies in or has a boundary point in common with $Q^{(n)}$. Therefore, in virtue of inequality (c),

$$g(f_\nu, PP') < \frac{2q^{(\nu-1)}}{q^{(n)}} = \frac{2q^{(\nu-1)}}{q^{(\nu-2)}} \cdots \frac{q^{(n+1)}}{q^{(n)}}$$

$$< 2\frac{1}{\rho_{\nu-1}} \cdots \frac{1}{\rho_{n+1}} < \frac{1}{2^{\nu-n-2}},$$

since $\rho_n > 2$ for $n > 1$. The reasoning here implies at first that $\nu > n+1$, but the final inequality is valid for $\nu = n+1$ also, and evidently, too, if either one or both of the points P and P' lie in F_ν. Hence

$$g(r_n, PP') < \sum_{\nu=n+1}^{\infty} \frac{1}{2^{\nu-n-2}} = 4.$$

Therefore, for $n \geqq 2$,

$$g(f, PP') \geqq g(s_n, PP') - g(r_n, PP') > n - 4.$$

Since P belongs to $\prod G_n$, a point P' satisfying the last inequality can be found for every positive integer $n \geqq 2$; and since PP', together with $q^{(n)}$ is, by (c), infinitesimal as $n \to \infty$, we conclude that the surface $z = f(x, y)$ is of unbounded grade at P.

Remark. The proof has been given for a surface $z = f(x, y)$ lying in a euclidean space, but the same reasoning applies to euclidean n-space. In fact, with certain modifications of the argument not hard to discern, our theorem, including the converse part, can be extended to any metric space, assumed, of course, if the theorem is to retain significance, to be without isolated points. The triangle postulate $d(P_1P_2) + d(P_2P_3) \geqq d(P_1P_3)$ for such a space turns out to be an adequate substitute for metric relations in the plane frequently utilized in the proof. For such an abstract space, we should, for example, change the squares to "spheres"; however, to show that $G^{(n)}$ is the sum of non-overlapping spheres $Q^{(n)}$, boundary included, we make use of Zermelo's Theorem on Normal Order.

Ohio State University