CONCERNING A SET OF AXIOMS FOR THE SEMI-QUADRATIC GEOMETRY OF A THREE-SPACE*  

BY J. L. DORROH  

In his paper *Sets of metrical hypotheses for geometry*, † R. L. Moore raises the question whether the set $O$ of order axioms and the set $C$ of congruence axioms employed therein, together with $M$, the proposition that every segment has a mid-point, and $P_2$, a form of the parallel postulate, are sufficient to give the semi-quadratic geometry of a three-space. At the same time, he states that this question may be answered in the affirmative if it can be proved on the basis of $O$, $C$, and $M$ that all right angles in space are congruent to each other. In the present paper it will be shown that $O$ and $C$ are sufficient to require that all right angles in space be congruent to each other.  

It is a result of a recent paper‡ of the present author that the theorems of sections 1, 2, 3, and 4 of M.H. are consequences of $O$ and $C$. Theorems from these sections of M.H. will be quoted without further mention of this justification of their use.

**Theorem 1.** If $A, B, C, D$ are four non-coplanar points such that $\angle ABD$ is a right angle§ and $\angle CBD$ is a right angle, and $E$ is any point distinct from $B$ and in the plane $ABC$, then $\angle EBD$ is a right angle.

**Proof.** If $E$ is a point of the line $AB$, or of the line $CB$, then, by hypothesis, $\angle EBD$ is a right angle.  

Suppose, then, that $E$ belongs to the plane $ABC$, is distinct from $B$, and belongs neither to the line $AB$ nor to the line $BC$. Let $C'$ denote a point such that $CBC'$. It follows by a corollary

---

* Presented to the Society, September 6, 1928.  
† Transactions of this Society, vol. 9 (1908), pp. 487–512. The notation M. H. will be used to designate this paper. Similarly, S. A. will be used to denote O. Veblen’s paper, *A system of axioms for geometry*, ibid., vol. 5 (1904), pp. 343–384.  
§ See Definition 7 of M. H., §3.
of Theorem 16 of S.A. that the line $BE$ contains a point $H$ such that $AHC$ or $AHC'$. Let $G$ denote one of the points $C$ or $C'$ so that $AHG$. Let $F$ denote a point such that $DBF$ and $DB \equiv BF$. Since by hypothesis the line $BD$ is perpendicular to the line $AB$ and to the line $BG$, it follows that $DG \equiv FG$ and $AD \equiv AF$. Since $AG \equiv AG$ and $AH \equiv AH$, it follows* that $DIH \equiv FH$. Hence, by definition, $\angle DBH$ is a right angle.

**Theorem 2.** If $L, M, N, O$ are four non-coplanar points such that $\angle LON$ is a right angle and $\angle MON$ is a right angle, then $\angle LON \equiv \angle MON$.

**Proof.** Since $L, M, N, O$ are non-coplanar, $L, O, M$ are non-collinear. Let $E$ denote a point such that the ray $OE$ bisects $\angle LOM$.† Let $M'$ denote a point in the order $MOM'$, and let $Q$ denote a point such that the ray $OQ$ bisects $\angle M'OL$. Then $\angle EOQ$ is a right angle.‡ Let $P$ denote a point such that $QOP$ and $OP \equiv OQ$; then $QE \equiv PE$. Also, since by Theorem 1 $\angle NOP \equiv \angle NOQ$, $QN \equiv PN$. The ray $OM$ contains a point $K$ such that $PKE$, and the ray $OL$ contains a point $R$ such that $QRE$. By Theorem 1 of M.H. §3, $OK \equiv OR$ and $EK \equiv ER$. Since $PKE$, $QRE$, $EP \equiv EQ$, $NE \equiv NE$, $NP \equiv NQ$, and $EK \equiv ER$, then $NK \equiv NR$,§ and, by definition, $\angle NOR \equiv \angle NOK$.

**Theorem 3.** If $\alpha_1$ and $\alpha_2$ are two intersecting planes and $\phi_1$ is a right angle in $\alpha_1$ and $\phi_2$ is a right angle in $\alpha_2$, then $\phi_1 \equiv \phi_2$.

**Proof.** Let $k$ denote the line of intersection|| of $\alpha_1$ and $\alpha_2$. Let $k_1$ denote a line in $\alpha_1$ perpendicular to $k$ at a point $O$ of $k$, and let $k_2$ denote a line in $\alpha_2$ perpendicular to $k$ at $O$. Let $\psi_1$ be a right angle formed by $k_1$ and $k$, and let $\psi_2$ be a right angle formed by $k_2$ and $k$. It follows from Theorem 2 that $\psi_1 \equiv \psi_2$.

* A special case of Theorem 11 of M. H. §1 may be stated as follows: If $A, B, C$ are three non-collinear points and $A', B', C'$ are three non-collinear points, and $ADC$, $A'D'C'$, $AB \equiv A'B'$, $AC \equiv A'C'$, $AD \equiv A'D'$, $BC \equiv B'C'$, then $BD \equiv B'D'$. For the suggestion that the figure used in the proof of Theorem 1 and the use of the particular theorem just stated would shorten the arguments I had previously given for Theorems 1 and 2, I am indebted to H. G. Forder.

† See a corollary of Theorem 6 of M. H., §3.

‡ See proof of Theorem 7 of M. H., §3.

§ See the theorem stated in a footnote on Theorem 1.

By Theorem 1 of M.H. §4, \( \phi_1 = \psi_1 \), and \( \phi_2 = \psi_2 \). It follows, then, from Theorem 14 of M.H. §1, that \( \phi_1 = \phi_2 \).

**Theorem 4.** If \( \phi_1 \) and \( \phi_2 \) are two right angles in space, then \( \phi_1 = \phi_2 \).

**Proof.** If \( \phi_1 \) and \( \phi_2 \) are in the same plane, \( \phi_1 = \phi_2 \) by Theorem 1 of M.H. §4. If \( \phi_1 \) and \( \phi_2 \) are not in the same plane, they lie in intersecting planes or in non-intersecting planes. If they lie in intersecting planes, they are congruent to each other by Theorem 3. If \( \phi_1 \) and \( \phi_2 \) lie in the planes \( \alpha_1 \) and \( \alpha_2 \), respectively, and \( \alpha_1 \) does not intersect \( \alpha_2 \), there exists a plane \( \alpha_3 \) which intersects both \( \alpha_1 \) and \( \alpha_2 \). There exists in \( \alpha_3 \) a right angle \( \phi_3 \). By Theorem 3, \( \phi_1 = \phi_3 \) and \( \phi_2 = \phi_3 \); hence, by Theorem 14 of M.H. §1, we have \( \phi_1 = \phi_2 \).

THE UNIVERSITY OF TEXAS

---

CERTAIN QUINARY FORMS RELATED TO THE SUM OF FIVE SQUARES*

BY B. W. JONES†

1. **Introduction.** The number of solutions in integers \( x, y, z \) of the equation \( n = x^2 + y^2 + z^2 \) is a function of the binary class number of \( n \). For numerous forms \( f = ax^2 + by^2 + cz^2 \), the expression of the number of solutions of \( f = n \) in terms of the class number is another way of showing that the number of representations of \( n \) by \( f \) is a function of the number of representations of various multiples of \( n \) as the sum of three squares.‡

Similarly, the number of solutions of the equation \( n = x^2 + y^2 + z^2 + t^2 \) in integers is the sum of the positive odd divisors of \( n \), multiplied by 8 or 24, according as \( n \) is odd or even. There are various forms \( f = ax^2 + by^2 + cz^2 + dt^2 \) for which the number of representations of \( n \) by \( f \) is a multiple of the sum of the odd divisors of \( n \). The number of representations of \( n \) by one of

---

* Presented to the Society, April 5, 1930.
† National Research Fellow.