By Theorem 1 of M.H. §4, \( \phi_1 = \psi_1 \), and \( \phi_2 = \psi_2 \). It follows, then, from Theorem 14 of M.H. §1, that \( \phi_1 = \phi_2 \).

**Theorem 4.** If \( \phi_1 \) and \( \phi_2 \) are two right angles in space, then \( \phi_1 = \phi_2 \).

**Proof.** If \( \phi_1 \) and \( \phi_2 \) are in the same plane, \( \phi_1 = \phi_2 \) by Theorem 1 of M.H. §4. If \( \phi_1 \) and \( \phi_2 \) are not in the same plane, they lie in intersecting planes or in non-intersecting planes. If they lie in intersecting planes, they are congruent to each other by Theorem 3. If \( \phi_1 \) and \( \phi_2 \) lie in the planes \( \alpha_1 \) and \( \alpha_2 \), respectively, and \( \alpha_1 \) does not intersect \( \alpha_2 \), there exists a plane \( \alpha_3 \) which intersects both \( \alpha_1 \) and \( \alpha_2 \). There exists in \( \alpha_3 \) a right angle \( \phi_3 \). By Theorem 3, \( \phi_1 = \phi_3 \) and \( \phi_2 = \phi_3 \); hence, by Theorem 14 of M.H. §1, we have \( \phi_1 = \phi_2 \).

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**CERTAIN QUINARY FORMS RELATED TO THE SUM OF FIVE SQUARES**

**BY B. W. JONES**

1. **Introduction.** The number of solutions in integers \( x, y, z \) of the equation \( n = x^2+y^2+z^2 \) is a function of the binary class number of \( n \). For numerous forms \( f = ax^2 + by^2 + cz^2 \), the expression of the number of solutions of \( f = n \) in terms of the class number is another way of showing that the number of representations of \( n \) by \( f \) is a function of the number of representations of various multiples of \( n \) as the sum of three squares.†

Similarly, the number of solutions of the equation \( n = x^2+y^2+z^2+t^2 \) in integers is the sum of the positive odd divisors of \( n \), multiplied by 8 or 24, according as \( n \) is odd or even. There are various forms \( f = ax^2 + by^2 + cz^2 + dt^2 \) for which the number of representations of \( n \) by \( f \) is a multiple of the sum of the odd divisors of \( n \). The number of representations of \( n \) by one of

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* Presented to the Society, April 5, 1930.
† National Research Fellow.
these forms is thus a simple function of the number of representations of \( n \) as the sum of four squares.*

Following a suggestion of E. T. Bell, I have here considered as the fundamental function, \( \phi(n) \), the number of representations of \( n \) as the sum of five squares. With the exception of three forms \( (f_{11}, f_{12}, f_{13}) \), the number of solutions of \( n=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2 \), where \( a_i = 1, 2 \) or 4, is shown to be expressible in terms of \( \phi \) and for \( a_i = 1, 2, 4, \) or 8 the number of solutions of \( f=n \) is expressed in terms of \( \phi \) and two other functions \( (\alpha \text{ and } \beta) \). It should be noted that for certain values of \( n, M_{abc}(n) \) is expressible totally in terms of \( \phi \). It is true in many cases when \( n \) is even and in the following formulas when \( n \) is odd: (37.2), (37.3), (38.1), (40.1), (40.2), (41.1), (44.1), (46.1), (48.1), (48.2), (50.1), (52.1). See also the last section of the paper giving a few miscellaneous results.

2. Notations. The letters \( n, m, x, y, \mu \) are used to denote integers; \( m \) and \( \mu \) are odd and \( n \) and \( m \) are positive.

\( N[n=f] \) denotes the number of representations of \( n \) by the form \( f=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2 \), the coefficients to be arranged in increasing order of magnitude.

\( f_j \) or \( f'_j \) is the form \( f \) when \( j \) of the coefficients are 2 or 4 respectively and the rest of the coefficients are 1.

\( f_{ij} \) is the form \( f \) when \( i \) of the coefficients are 2, \( j \) of them 4 and the remainder are 1.

\( f_{abc} \) is the form \( x_1^2+ax_2^2+bx_3^2+cx_4^2+8x_5^2 \) where \( a, b \) and \( c \) are powers of 2.

\( M_j(n)=N[n=f_j] \); \( M'_j(n)=N[n=f'_j] \); \( M_{ij}(n)=N[n=f_{ij}] \);

\( M_{abc}(n)=N[n=f_{abc}] \).

We regard the following as fundamental functions:

\( M_0(n)=\phi(n) \); \( \alpha(m)=M_{11}(m) \); \( \beta(m)=M_{244}(m) \), if \( m \equiv 1 \) (mod 8).

We also use the following for brevity's sake: \( \lambda(n)=M'_1(n) \); \( \lambda'(4n)=N[4n=f'_1; x_1x_2x_3x_4 \text{ odd}] \); \( \alpha'(n)=N[n=f_{11} \text{ with } x_1 \text{ odd}] \) and \( \phi'(m)=N[m=f_0; x_1x_2x_3x_4 \text{ odd}] \), which has a value different from 0 only when \( m \equiv 5 \) (mod 8).

3. A Fundamental Lemma.† \( N[2n=x^2+y^2]=N[n=x^2+y^2] \).

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* See, for example, J. Liouville, Journal de Mathématiques, (2), vol. 7 (1862); P. Pepin, Journal de Mathématiques, (4), vol. 6 (1890), p. 5.

† Since this lemma is very elementary, we shall use it freely without comment.
This follows from the fact that \( 2n = x^2 + y^2 \) implies that the pair of equations \( x + y = 2X \) and \( x - y = 2Y \) is solvable for \( X \) and \( Y \) and there is a one to one correspondence between the solutions of \( 2n = x^2 + y^2 \) and \( n = X^2 + Y^2 \).

**Corollary 1.**

\[ N[2m = x^2 + y^2] = 2N[m = \mu^2 + 4y^2]. \]

**Corollary 2.**

\[ N[2n' = x^2 + y^2] = N[n' = x^2 + y^2] = N[2n' = 2x^2 + 2y^2]. \]

4. **Reduction Formulas for \( \phi(n) \).** Since \( f_0 = n \equiv 0 \pmod{4} \) implies that just one or all of the \( x \)'s are even, we have

\[ \phi(4n) - \phi(n) = 5\lambda'(4n). \]

Applying Corollary 1, we have

\[ \lambda'(4n) = 4N[2n = \mu^2 + \mu_x^2 + 4x_3^2 + 4x_4^2 + 2x_5^2] = 8N[n = \mu^2 + 4x_2^2 + 2x_3^2 + 2x_4 + x_5^2]. \]

Applying Corollary 2, we have

\[ \lambda'(4n) = 8\lambda'(n) \text{ if } n \equiv 0 \pmod{4}. \]

If \( n = 2m \) note that \( f_0 = 2m \) implies that exactly two \( x \)'s are odd and we have from \((1')\)

\[ \lambda'(8m) = 4\phi(2m)/5. \]

If \( n = m \) consider first the case \( m \equiv 5 \pmod{8} \). Then

\[ \phi'(m) = 4N[m = f_{228} \text{ with } x_1x_2x_3 \text{ odd}] = 8N[m = \mu^2 + 4\mu_x^2 + 8x_3^2 + 16x_4^2 + 8x_5^2]. \]

Now \( f'_4 = m \) implies that one of \( x_2, x_3, x_4, x_5 \) is incongruent mod 2 to the other three, and thus

\[ M'_4(m) = 4N[m = f'_4 \text{ ; } x_2 \not\equiv x_3 \equiv x_4 \equiv x_5 \pmod{2}] \]
\[ = 4N[m = \mu_1^2 + 4\mu_2^2 + 16x_3^2 + 8x_4^2 + 8x_5^2; x_4 \equiv x_5 \pmod{2}] \]
\[ + 4N[m = \mu_1^2 + 16x_3^2 + 4\mu_2^2 + 8x_4^2 + 8x_5^2; x_4 \not\equiv x_5 \pmod{2}] \]
\[ = 4N[m = \mu_1^2 + 4\mu_2^2 + 8x_3^2 + 16x_4^2 + 8x_5^2]. \]

Thus \( M'_4(m) = \phi'(m)/2. \) This taken with the known equation
$5M_4'(m)+\phi'(m)=\phi(m)$, found by noting that $f_0=m$ implies that just one or all the $x$'s are odd, gives

(4) \[ \phi'(m)2\phi(m)/7 \text{ if } m \equiv 5 \pmod{8}, \]

and

(5) \[ M_4'(m) = \phi(m)/7 \text{ if } m \equiv 5 \pmod{8}. \]

Now from (1'), we have $\lambda'(4m)=8M_4'(m)$ if $m \equiv 5 \pmod{8}$. Since $f_0=m$ implies that just three of the $x$'s are odd, or just one is odd according as $m \equiv 3 \pmod{4}$ or $\equiv 1 \pmod{8}$, we have, from (1') and (5),

(6) \[ \lambda'(4m) = 8a\phi(m), \]

where $a = 1/10, 1/5, \text{ or } 1/7$, according as

\[ m \equiv 3 \pmod{4}, \equiv 1 \pmod{8}, \text{ or } \equiv 5 \pmod{8}. \]

If $n' \not\equiv 0 \pmod{4}$, we have, using (1) and (2),

\( \phi(4\alpha n') - \phi(4\alpha^{-1}n') = 5 \cdot 8^{\alpha-1}\lambda'(4n'), (\alpha \geq 1), \)
\( \phi(4\alpha^{-1}n') - \phi(4\alpha^{-2}n') = 5 \cdot 8^{\alpha-2}\lambda'(4n'), (\alpha \geq 2), \)

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\( \phi(4n') - \phi(n') = 5\lambda'(4n'). \)

Adding, we get

(7) \[ \phi(4\alpha n') - \phi(n') = 5^{\frac{8\alpha - 1}{7}} \lambda'(4n'), \]

where $\alpha \geq 1$ and $n' \not\equiv 0 \pmod{4}$. Then, using (3) and (6), we have the reduction formulas

(8.1) \[ \phi(2^{2\alpha+1}m) = (2^{3\alpha+2} + 3)\phi(2m)/7, \]

for $\alpha \geq 1$,

(8.2) \[ \phi(4\alpha m) = A\phi(m)/7, \]

where $\alpha \geq 1$ and $A = 2^{2\alpha+2} + 3, 8^{\alpha+1} - 1, (5 \cdot 8^{\alpha+2} + 9)/7$ according as $m \equiv 3 \pmod{4}, \equiv 1 \pmod{8}$, or $\equiv 5 \pmod{8}$.

5. Relationship between $\lambda(n)$ and $\phi(n)$. It is obvious that $\lambda'(4n) = \lambda(4n) -\phi(n)$. Thus, from (1), we have $\phi(4n) + 4\phi(n) = 5\lambda(4n)$. This from (8) gives the formulas below for $\lambda(4n)$. We
find $\lambda(m)$ by noting that $f_0 = m$ implies that just three of the $x$’s are odd or just one is odd according as $m \equiv 3$ (mod 4) or $m \equiv 1$ (mod 8) and that $\lambda(m) = 4M_4'(m)$ if $m \equiv 5$ (mod 8). To obtain $\lambda(2m)$ note that $f_0 = 2m$ implies just two $x$’s are odd.

(9.1) \[ \lambda(4^{a+1}m) = B\phi(m), \]
where $\alpha \geq 0$ and $B = 3(2^{3a+4}+5)/35$, $(3\cdot 2^{3a+6} - 5)/35$, or $3(2^{3a+5}+3)/49$, according as $m \equiv 3$ (mod 4), $\equiv 1$ (mod 8), or $\equiv 5$ (mod 8);

(9.2) \[ \lambda(4^a\cdot 2m) = 3(2^{3a+1} + 5)\phi(2m)/35, \]
where $\alpha \geq 0$;

(9.3) \[ \lambda(m) = 4\alpha\phi(m), \]
where $\alpha$ is defined in (6).

6. Forms $f$, where $M(n)$ is Expressible Totally in Terms of $\phi$.*

CASE I: $n = m$. Note that $\lambda(2m) = 6N[f_1' = 2m; x_1, x_2$ odd and both $x_3$ and $x_4$ even$] = 12\alpha'(m)$. Thus

(10) \[ \alpha'(m) = \phi(2m)/20. \]

Now $f_i = m$ implies that one of $x_1, x_2, x_3, x_4$ is incongruent to the other three modulo 2, that is,

$M_1(m) = 4N[m = f_1; x_1 \neq x_2 \equiv x_3 \equiv x_4 (mod 2)] = 4\alpha'(m)$

and

(11) \[ M_1(m) = \phi(2m)/5. \]

The equation $f_i' = m$ implies that just three of $x_1, x_2, x_3, x_4$ are odd, or just one is odd, according as $m \equiv 3$ or 1 (mod 4). Thus, using (9.3),

(12) \[ M_4'(m) = b\phi(m), \] where $b = 1/10, 3/5, 3/7,$

according as $m \equiv 3$ (mod 4), $\equiv 1$ (mod 8), or $\equiv 5$ (mod 8).

We note that

$M_4(m) = N[m = f_0; x_1$ odd, $x_2 \equiv x_3, x_4 \equiv x_5 (mod 2)].$

It is therefore true that $M_4(m) = 2M_4'(m), M_4'(m)/3$ or

* For complete results see case II below.
$M_4'(m)/3 + \phi'(m)$, according as $m \equiv 3 \pmod{4}$, $\equiv 1 \pmod{8}$, or $\equiv 5 \pmod{8}$. Thus, from (12) and (4),

\begin{equation}
M_4(m) = c\phi(m),
\end{equation}

where $c = 3/7$ or $1/5$ according as $m \equiv 5 \pmod{8}$ or $\not\equiv 5 \pmod{8}$.

Also $m = f_2'$ implies that all of $x_1$, $x_2$, $x_3$ are odd, or just one is odd according as $m \equiv 3$ or $1 \pmod{4}$ and thus $M_{22}(m) = M_2'(m)$ or $M_2'(m)/3$ respectively. We have

\begin{equation}
M_{22}(m) = a\phi(m),
\end{equation}

where $a$ is defined in (6). Since $f_2 = m$ implies that just one of $x_1$, $x_2$, $x_3$ is odd, or all are odd, we see that

\[M_2(m) = N'(m) + 3M_{22}(m),\]

where $N'(m) = N[m = f_2; x_1x_2x_3$ odd]. Now

\[N'(m) = M_2'(m), 0, \text{ or } \phi'(m),\]

according as $m \equiv 3 \pmod{4}$, $\equiv 1$, or $5 \pmod{8}$. Using (12), (14) and (4), we have

\begin{equation}
M_2(m) = d\phi(m),
\end{equation}

where $d = 2/5$, $3/5$, or $5/7$, according as $m \equiv 3 \pmod{4}$, $\equiv 1 \pmod{8}$, or $\equiv 5 \pmod{8}$.

The following results are obvious:

\begin{align*}
&\text{(16)} \quad M_{31}(m) = \alpha'(m) = \phi(2m)/20; \\
&\text{(17)} \quad M_3(m) = 2M_{31}(m) = \phi(2m)/10; \\
&\text{(18.1)} \quad M_4'(m) = 2M_2'(m)/3 = 2\phi(m)/5 \text{ or } 2\phi(m)/7,
\end{align*}

according as $m \equiv 1$ or $5 \pmod{8}$;

\begin{align*}
&\text{(18.2)} \quad M_4'(4m + 3) = 0; \\
&\text{(19)} \quad M_4'(m) = \frac{1}{3}M_4'(m) = e\phi(m),
\end{align*}

where $e = 0, 1/5$, or $1/7$ according as $m \equiv 3 \pmod{4}$, $\equiv 1 \pmod{8}$, or $\equiv 5 \pmod{8}$;

\begin{equation}
M_{21}(m) = 2M_{22}(m) = 2a\phi(m),
\end{equation}

where $a$ is defined in (6).

**Case II: n even.** We express $N[n = f]$ in terms of $\lambda$ and $\phi$ from which, by reference to formulas (9) and (8), $N[n = f]$ may
be expressed in terms of $\phi$ alone. Since $f_1' = 4n$ implies $x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod{2},$

\[(21) \quad M_1(2n) = \lambda(4n), \quad \text{(for } M_1(m) \text{ see (11)),}\]

\[(22) \quad M_3(2n) = \phi(n), \quad \text{(for } M_3(m) \text{ see (17)).}\]

Since

\[\phi(2n) = N[4n = f_3; x_1 \equiv x_2 \pmod{2}] = M_2(2n) + 2N'(2n),\]

where $N'(2n) = N[2n = f_1'; x_1 \text{ odd}],$ we see that

\[N'(2m) = 3N[2m = f_2'] = 6M_{31}(m),\]

and that

\[N'(4n) = \lambda'(4n).\]

Using (16) and (1), we then have

\[(23.1) \quad M_2(2m) = 2\phi(2m)/5, \quad \text{(for } M_2(m) \text{ see (15)),}\]

\[(23.2) \quad M_2(4n) = \phi(4n) - 2N'(4n) = \{3\phi(4n) + 2\phi(n)\}/5.\]

Obviously,

\[(24) \quad M_4(2n) = M_1(n) = \phi(2n)/5 \text{ or } \lambda(2n),\]

according as $n$ is odd or even. (For $M_4(m)$ see (13).)

Now $M_2'(2m) = 3N[2m = f_2'] = 6M_{31}(m).$ Thus, using (16), we get

\[(25.1) \quad M_2'(2m) = 3\phi(2m)/10.\]

For $M_2'(m),$ see (12).

\[(25.2) \quad M_2'(4n) = \phi(n).\]

Also $M_2'(2n) = M_3(n),$ using (17),

\[(26) \quad M_3'(2n) = \phi(2n)/10 \text{ or } \phi(n)/2,\]

according as $n$ is odd or even. (For $M_3'(m),$ see (18).)

The following results are obvious:

\[(27) \quad M_4'(2n) = 0 \text{ or } \phi(n/2),\]

according as $n$ is odd or even. (For $M_4'(m)$ see (19).)

\[(28) \quad M_{21}(2n) = M_4(n) = \phi(2n)/5 \text{ or } \lambda(2n),\]

according as $n$ is odd or even. (For $M_{21}(m),$ see (20).)
(29) \[ M_{22}(2n) = M_3(n) = \phi(2n)/10 \text{ or } \phi(n/2), \]
according as \( n \) is odd or even. (For \( M_{22}(m) \), see (14).)
(30.1) \[ M_{31}(2m) = M_2(m) = d\phi(m), \]
where \( d \) is defined in (15).
(30.2) \[ M_{31}(4n) = 2\phi(2n)/5 \text{ or } \{3\phi(2n) + 2\phi(n/2)\}/5, \]
according as \( n \) is odd or even. (For \( M_{31}(m) \), see (16).)

7. A Reduction Formula for \( \alpha'(n) \). A reduction formula for 
\( \alpha'(n) \) will later be found necessary. We see that
\[ \alpha'(2m) = 2N[2m = \mu_1^2 + \mu_2^2 + 8x_3^2 + 4x_4^2 + 4x_5^2] = 4M_{22}(m) \]
and
\[ \alpha'(4n) = 2N[2n = \mu_1^2 + \mu_2^2 + 2\mu_3^2 + 4x_4^2 + 4x_5^2] \]
\[ = 4N[2n = f_{31}; x_1x_2 \text{ odd}] = 8\alpha'(n). \]
Thus, using (10), we have
(31.1) \[ \alpha'(4a_m) = 8\alpha\phi(2m)/20, \alpha \geq 0, \]
(31.2) \[ \alpha'(4^a . 2m) = 4 \cdot 8\alpha\phi(2m), \]
where \( a \) is defined in (6) and \( \alpha \geq 0. \)

8. \( M_{11}, M_{13}, M_{12} \) Expressed in Terms of \( \alpha \) and \( \phi \).* It is clear
that \( M_{11}(2n) = 6N'(n) + M_4(n) \) where \( N'(n) = N[n = f_{21}; x_1 \text{ odd}] \)
\[ = M_{22}(n) \text{ or } 2\alpha'(n/2) \text{ according as } n \text{ is odd or even, and} \]
(32.1) \[ M_{11}(m) = \alpha(m), \text{ by definition,} \]
(32.2) \[ M_{11}(2m) = g\phi(m), \text{ where } g = 4/5, 7/5, \text{ or } 9/7, \]
according as \( m \equiv 3 \pmod{4}, \equiv 1 \pmod{8}, \text{ or } \equiv 5 \pmod{8}; \)
(32.3) \[ M_{11}(4n) = 12\alpha'(n) + M_4(2n). \]
Now
\[ \alpha(m) = N[m = f_{11}; x_2 \equiv x_3 \pmod{2}] \]
\[ + 2N[m = f_{12}; x_2 \equiv 1 \pmod{2}] = \alpha'(m) + 2M_{13}(m), \]

*In many cases, to save space, results are expressed in terms of \( M_5, M_7, M_{11}, M_{21}, M_{23}, M_{31}, \alpha' \) which have been previously expressed in terms of \( \phi \).
(33.1) \( M_{13}(m) = \frac{1}{2} \{ \alpha(m) - \alpha'(m) \} = \alpha(m)/2 - \phi(2m)/40, \)
(33.2) \( M_{13}(2n) = M_4(n), \)
(34.1) \( M_{12}(m) = 2M_{13}(m) = \alpha(m) - \phi(2m)/20, \)
(34.2) \( M_{12}(2n) = M_2(n). \)

9. \( M_{abc} \) Expressed in Terms of \( \alpha, \beta' \) and \( \phi. \)

If \( n = m \equiv 1 \pmod{4}, \)
\[ M_{13}(m) = M_{444}(m) = M_{122}(m)/2. \]

If \( n = m \equiv 3 \pmod{4}, \)
\[ M_1(m) = 4N \left[ m = f_{11}; x_1x_2x_3 \text{ odd} \right] + 4M_{13}(m) \]
\[ = 8N \left[ m = f_{224}; x_1x_2 \text{ odd} \right] + 4M_{13}(m) \]
\[ = 8M_{248}(m) + 4M_{13}(m) = 2M_{122}(m) + 4M_{13}(m). \]

(35.1) \( M_{122}(m) = \alpha(m) - \phi(2m)/20 \) if \( m \equiv 1 \pmod{4}, \)
(35.2) \( M_{122}(m) = 3\phi(2m)/20 - \alpha(m) \) if \( m \equiv 3 \pmod{4}, \)
(35.3) \( M_{122}(2n) = \lambda(n). \)

Noting that \( M_{111}(m) = 4M_{224}(m) = 2M_{122}(m), \) we find

(36.1) \( M_{111}(m) = 2\alpha(m) - \phi(2m)/10, \)
if \( m \equiv 1 \pmod{4}; \)
(36.2) \( M_{111}(m) = 3\phi(2m)/10 - 2\alpha(m), \)
if \( m \equiv 3 \pmod{4}; \)
(36.3) \( M_{111}(2m) = 6M_{21}(m) = 12a\phi(m), \)
where \( a \) is defined in (6);
(36.4) \( M_{111}(4n) = \lambda(2n). \)

If \( m \equiv 5 \pmod{8}, \)
\[ M_{448}(m) = N \left[ m = f_4'; x_4 \equiv x_6 \pmod{2} \right] \]
and
\[ 4N \left[ m = f_4' \neq x_2 \equiv x_4 \equiv x_6 \pmod{2} \right] \]
\[ = 4N \left[ m = x_1^2 + 4x_2^2 + 16x_3^2 + 4x_4^2 + 4x_5^2 \right] = 2M_{244}(m). \]

* See note on §8.
Then

(37.1) \[ M_{244}(m) = \beta(m), \]

if \( m \equiv 1 \pmod{8} \) by definition;

(37.2) \[ M_{244}(m) = M_{4}(m)/2 = \phi(m)/14. \]

if \( m \equiv 5 \pmod{8} \);

(37.3) \[ M_{244}(m) = M_{22}(m)/2 = \phi(m)/20 \]

if \( m \equiv 3 \pmod{4} \);

(37.4) \[ M_{244}(2n) = M_{31}(n). \]

It is clear that

\[ M_{112}(m) = 3M_{244}(m) + N(m), \]

where \( N(m) = N[m = f_{112}; x_{1}x_{2}x_{3} \text{ odd}] \).

Now

\[ N(m) = 2N[m = \mu_{1}^{2} + 2\mu_{2}^{2} + 8x_{3}^{2} + 2x_{4}^{2} + 8x_{5}^{2} = M_{22}(m), \]
\[ \phi(m)/2, \text{ or 0, according as } m \equiv 3, 5, \text{ or } \pm 1 \pmod{8}. \]

Moreover,

\[ M_{112}(2n) = M_{31}(n) + 3N[2n = f_{124}; x_{1}x_{2} \text{ odd}] = M_{31}(n) + 6N''(n) \]

where \( N''(n) = N[n = f_{12}; x_{1} \text{ odd}] \). Now \( N''(n) = M_{13}(n) \) if \( n \) is odd, 2\( M_{22}(n)/2 \) if \( n \equiv 2 \pmod{4} \), or 4\( \alpha'(n/4) \) if \( n \equiv 0 \pmod{4} \).

(38.1) \[ M_{112}(m) = 3\beta(m), \ 5\phi(m)/14, \ \phi(m)/4, \text{ or } 3\phi(m)/20, \]

according as \( m \equiv 1, 5, 3, \text{ or } 7 \pmod{8} \);

(38.2) \[ M_{112}(2m) = 3\alpha(m) - \phi(2m)/10; \]

(38.3) \[ M_{112}(4m) = (d + 12a)\phi(m), \]

where \( d \) and \( a \) are defined in (15) and (6), respectively;

(38.4) \[ M_{112}(8n) = M_{31}(4n) + 24\alpha'(n); \]

(39.1) \[ M_{114}(m) = 3M_{111}(m)/4 = 3\alpha(m)/2 - 3\phi(2m)/40 \text{ if } \]
\[ m \equiv 1 \pmod{4}; \]

(39.2) \[ M_{114}(m) = M_{111}(m)/4 = 3\phi(2m)/40 - \alpha(m)/2 \text{ if } \]
\[ m \equiv 3 \pmod{4}; \]

(39.3) \[ M_{114}(2m) = 3M_{21}(m) = 6a\phi(m), \]

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where \( a \) is defined in (6);

(39.4) \[ M_{114}(4n) = M_1(n). \]

Noting that \( M_{118}(m) = 2N[m = f_{288}; x_2 \text{ odd}] + 3M_{448}(m) \), we have

(40.1) \[ M_{118}(m) = 2M_{244}(m) = \phi(m)/10, \]
if \( m \equiv 3 \) (mod 8);

(40.2) \[ M_{118}(m) = 3M_{244}(m) = 3\beta(m) \text{ or } 3\phi(m)/14, \]
according as \( m \equiv 1 \) or 5 (mod 8);

(40.3) \[ M_{118}(m) = 0 \] if \( m \equiv 7 \) (mod 8);

(40.4) \[ M_{118}(2m) = 3M_{12}(m) = 3\alpha(m) - 3\phi(2m)/20; \]

(40.5) \[ M_{118}(4n) = M_2(n). \]

Noting that \( M_{124}(m) = 2M_{244}(m) \), we have,

(41.1) \[ M_{124}(m) = 2\beta(m), \phi(m)/7, \text{ or } \phi(m)/10, \]
according as \( m \equiv 1 \) (mod 8), \( \equiv 5 \) (mod 8), or \( \equiv 3 \) (mod 4);

(41.2) \[ M_{124}(2m) = \alpha(m); \]

(41.3) \[ M_{124}(4m) = g\phi(m), \]
where \( g \) is defined in (32.2);

(41.4) \[ M_{124}(8n) = 12\alpha'(n) + M_4(2n). \]

Now \( M_{128}(m) = M_{122}(m)/2 \) if \( m \equiv 3 \) (mod 4). But if \( m \equiv 1 \) (mod 4),

\[ M_1(m) = N[m = f_1; \text{ just three of } x_1, x_2, x_3, x_4 \text{ odd}] \]
\[ + N[m = f_1; \text{ just one of } x_1, x_2, x_3, x_4 \text{ odd}] \]
\[ = 8N[m = f_{224}; x_1x_2x_3 \text{ odd}] + 4M_{13}(m). \]

Also

\[ M_{122}(m) = M_{188}(m) + 2N[m = f_{224}; x_1x_2x_3 \text{ odd}]. \]

Therefore

\[ M_1(m) - 4M_{122}(m) = 4M_{13}(m) - 4M_{128}(m); \]

and

(42.1) \[ M_{128}(m) = 3\phi(2m)/40 - \alpha(m)/2, \]
if \( m \equiv 3 \pmod{4} \);

(42.2) \( M_{128}(m) = 3\alpha(m)/2 - \phi(2m)/8 \),

if \( m \equiv 1 \pmod{4} \);

(42.3) \( M_{128}(2n) = M'_n \).

Since \( M_{144}(m) = M_{12}(m) \), if \( m \equiv 1 \pmod{4} \), we have

(43.1) \( M_{144}(m) = \alpha(m) - \phi(2m)/20 \) or 0,

according as \( m \equiv 1 \) or 3 \( \pmod{4} \);

(43.2) \( M_{144}(2n) = M_{21}(n) \);

(44.1) \( M_{148}(m) = 2M_{244}(m) = 2\beta(m) \) or \( \phi(m)/7 \),

according as \( m \equiv 1 \) or 5 \( \pmod{8} \);

(44.2) \( M_{148}(m) = 0 \) if \( m \equiv 3 \pmod{4} \);

(44.3) \( M_{148}(2m) = M_{12}(m) = \alpha(m) - \phi(2m)/20 \);

(44.4) \( M_{148}(4n) = M_4(n) \);

(45.1) \( M_{188}(m) = M_{128}(m) = 3\alpha(m)/2 - \phi(2m)/8 \),

if \( m \equiv 1 \pmod{4} \);

(45.2) \( M_{188}(m) = 0 \) if \( m \equiv 3 \pmod{4} \);

(45.3) \( M_{188}(2n) = M'_n \);

(46.1) \( M_{222}(m) = M_{112}(m)/3, 3M_4(m)/4, M_4(m)/2, \) or \( M_4(m)/4, \)

that is, \( \beta(m), 3\phi(m)/20, 3\phi(m)/14 \) or \( \phi(m)/20 \), according as \( m \equiv 1, 3, 5, \) or 7 \( \pmod{8} \);

(46.2) \( M_{222}(2m) = \alpha(m) \);

(46.3) \( M_{222}(4m) = 8\phi(m) \),

where \( g \) is defined in (32.2);

(46.4) \( M_{222}(8n) = 12\alpha'(n) + M_4(2n) \);

(47.1) \( M_{224}(m) = M_{122}(m)/2 = \alpha(m)/2 - \phi(2m)/40 \),

if \( m \equiv 1 \pmod{4} \);

(47.2) \( M_{224}(m) = 3\phi(2m)/40 - \alpha(m)/2 \),

if \( m \equiv 3 \pmod{4} \);
(47.3) \( M_{22s}(2n) = M_{21}(n) \).

Since \( M_{22s}(m) = M_{2s4}(m) \) or \( 2M_{2s4}(m) \), according as \( m \equiv 1 \) (mod 4) or 3 (mod 8),

(48.1) \( M_{22s}(m) = \beta(m) \) or \( \phi(m)/14 \),
according as \( m \equiv 1 \) or 5 (mod 8);

(48.2) \( M_{22s}(m) = \phi(m)/10 \) or 0,
according as \( m \equiv 3 \) or 7 (mod 8);

(48.3) \( M_{22s}(2m) = M_{12}(m) = \alpha(m) - \phi(2m)/20 \);

(48.4) \( M_{22s}(4n) = M_{4}(n) \);

(49.1) \( M_{248}(m) = M_{128}(m)/2 = 3\alpha(m)/4 - \phi(2m)/16 \),
if \( m \equiv 1 \) (mod 4);

(49.2) \( M_{248}(m) = 3\phi(2m)/80 - \alpha(m)/4 \),
if \( m \equiv 3 \) (mod 4);

(49.3) \( M_{248}(2n) = M_{22}(n) \).

Since \( M_{22s}(m) = M_{22s}(m)/3 \) or \( M_{22s}(m) \), according as \( m \equiv 3 \) or 1 (mod 8);

(50.1) \( M_{2ss}(m) = \beta(m), \phi(m)/20, 0, \) or 0,
according as \( m \equiv 1, 3, 5, \) or 7 (mod 8);

(50.2) \( M_{2ss}(2m) = M_{13}(m) = \alpha(m)/2 - \phi(2m)/40 \);

(50.3) \( M_{2ss}(4n) = M_{4}(n) \);

(51.1) \( M_{444}(m) = M_{13}(m) = \alpha(m)/2 - \phi(2m)/40 \)
if \( m \equiv 1 \) (mod 4);

(51.2) \( M_{444}(n) = 0 \) if \( n \equiv 2 \) or 3 (mod 4);

(51.3) \( M_{444}(4n) = M_{3}(n) \).

Since \( M_{448}(m) = M_{148}(m)/2 \) if \( m \equiv 1 \) (mod 4), we have

(52.1) \( M_{448}(m) = \beta(m) \) or \( \phi(m)/14 \),
according as \( m \equiv 1 \) or 5 (mod 8);

(52.2) \( M_{448}(n) = 0 \) if \( n \equiv 2 \) or 3 (mod 4);
(52.3) \( M_{448}(4n) = M_2(n) \);
(53.1) \( M_{488}(m) = M_{248}(m) = 3\alpha(m)/4 - \phi(2m)/16 \),
if \( m \equiv 1 \pmod{4} \);
(53.2) \( M_{488}(n) = 0 \) if \( n \equiv 2 \) or 3 \( \pmod{4} \);
(53.3) \( M_{488}(4n) = M_3(n) \);
(54.1) \( M_{888}(m) = M_{288}(m) = \beta(m) \),
if \( m \equiv 1 \pmod{8} \);
(54.2) \( M_{888}(n) = 0 \),
if \( n \equiv 2 \) or 3 \( \pmod{4} \) or 5 \( \pmod{8} \);
(54.3) \( M_{888}(4n) = M_4(n) \).

10. Some Miscellaneous Results. Letting \( f'_{abc} \) denote the form

\[ x_1^2 + ax_2^2 + bx_3^2 + cx_4^2 + 16x_5^2 \]

and

\[ M'_{abc}(n) = N \left[ f'_{abc} = n \right], \]

we have

(55) \( M'_{114}(8n + 7) = M'_2(8n + 7)/2 = \phi(8n + 7)/20 \);
(56.1) \( M'_{111}(8n + 7) = 4M'_{114}(8n + 7) = \phi(8n + 7)/5 \);
(56.2) \( M'_{111}(4m) = 8M_{22}(m) + \lambda(m) = 12a\phi(m) \),

where \( a \) is defined in (6);
(56.3) \( M'_{111}(8n) = \lambda(2n) \);
(57.1) \( M'_{248}(8n + 7) = M_{248}(8n + 7)/2 = \phi(8n + 7)/40 \);
(57.2) \( M'_{248}(8n + 5) = M_{448}(8n + 5)/2 = \phi(8n + 5)/28 \);
(58) \( M'_{488}(8n + 5) = M'_{248}(8n + 5) = \phi(8n + 5)/28 \).

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