A SERIES OF RATIONAL SURFACES

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Montesano* has considered a rational sextic surface $\sum_6$, whose plane system is the web of nonics having 8 fixed triple points and 3 simple points. The double curve is composed of a triple line and 3 double lines. The latter are coplanar and concurrent at a point on the triple line, the intersection being a quadruple point of the surface. This surface is one of a series whose orders are successive multiples of three. The double curve in each case is composed of a k-fold line, of multiplicity 3 less than the order of the surface, and k double lines which all meet the k-fold line, but which, except in the $\sum_6$ just mentioned, are skew to each other. From each surface may be derived by quadratic transformation another of degree one lower, whose double curve consists of a $(k-1)$-fold line and $k-2$ double lines, and which has an isolated triple point. The quintic so obtained from $\sum_6$ is exceptional in that it has 2 consecutive skew double lines (instead of 1), and its triple point is on the double line. I have described that surface in a previous paper.†

A curve of order $3r$, $r \geqq 3$, having 8 fixed r-fold points, $A_1, \cdots, A_8$, an $(r-3)$-fold point at A_0, the ninth base point of the pencil of cubics determined by $A_1, \cdots, A_8$, and $3(r-2)$ fixed simple points has 3 degrees of freedom. Two curves of this web have $3(r-1)$ variable intersections. Such a web defines a rational surface whose plane sections correspond to the curves of the web. For $r=2$ we have indeed the web of sextics with 8 fixed double points. But this case is altogether exceptional, since any two curves of this web which pass through an arbitrary point pass also through another point determined by the first. The corresponding surface is in fact a quadric cone. It will be convenient to consider independently the case $r=4$. There are ∞^4 curves of order 12 that have in common 8 fixed quadruple points and 6 fixed simple points. There

* Montesano, Rendiconti di Napoli, vol. 46 (1907), p. 66.

† This Bulletin, vol. 34 (1928), p. 761.

is a single nonic c_9 having triple points at the 8 points $A_1, \cdots, A_8$ and simple points at the 6 points $B_1, \cdots, B_6$. Corresponding to this family of duodecimics we have in 4-space a surface of order 10, whose hyperplane sections correspond to the curves of the family. One pencil of hyperplane sections correspond to the duodecimics composed of the nonic c_9 and the pencil of cubics through $A_1, \cdots, A_8$. Since the cubics of this pencil have 4 variable intersections with the duodecimics of the family, this pencil of hyperplane sections have in common a sextic curve with 6 nodes lying in a plane π and a point A_0' in π, but not the sextic, which corresponds to A_0, the ninth base point of the pencil of cubics. Similarly there is a pencil of hyperplane sections corresponding to the duodecimics made up of the cubic c_3^i, determined by B_i, and the pencil of nonics n_i, which have the 8 points A for triple points and pass through the points B, except B_i. To the cubic c_3^i corresponds on the surface a cubic lying in a plane σ_i. The hyperplane section corresponding to c_3^i and c_9 consists of the sextic in π, the cubic in σ_i, and the line corresponding to B_i. This hyperplane section belongs to both pencils and hence π and σ_i have a line s_i in common which passes through A_0'. This line meets the plane sextic in π in two points corresponding to the points in which $c_3{}^i$ meets c_9, and in 4 more points which correspond to the 4 remaining base points of the pencil of nonics n_i. To a nonic of this pencil corresponds a skew septimic which lies in a hyperplane containing σ_i, and meets σ_i in the 4-points where s_i meets the sextic in π, and in 3 variable points (not in π) corresponding to the intersections of the nonic with the cubic $c_3{}^i$. To a general cubic through the 8 points A corresponds a skew quartic which lies in a hyperplane containing π, and which meets π in A_0' and 3 variable points that correspond to the intersections of the cubic with c_9. We have thus in 4 space a plane π containing a sextic belonging to the surface, and 6 planes σ each containing a cubic belonging to the surface. The 6 planes σ meet π in 6 lines which have in common A_0'. The cubic in each plane σ passes through A_0'.

If now we project the surface from A_0' upon a hyperplane S_3 we obtain a surface $\sum_9$, of order 9, having a 6-fold line k, the trace of π in S_3, and 6 double lines, $d_1, \cdots, d_6$, all of which meet the 6-fold line, but are skew. There are 6 simple lines, the projections of the lines corresponding to the points B, each lying in a plane with the 6-fold line and one of the double lines. The tangent plane at A_0' gives another line l_0 on the surface in S_3, that meets the 6 double lines, but not the 6-fold line. This configuration in 4-space seemed worthy of notice. The result of the projection might of course have been obtained directly by use of the web of duodecimics having quadruple points at the 8 points A, simple points at the 6 points B, and a simple point at A_0. To the 6-fold line corresponds the nonic c_9 having triple points at $A_1, \cdots, A_8$ and simple points at $B_1, \cdots, B_6$. To a double line corresponds a cubic of the pencil determined by a point B. The genus of a plane section of $\sum_9$ is 7, and is accounted for by the 6-fold line and the 6 double lines. The class of the surface is $3(12-1)^2$ less 39 for each quadruple point, that is, 51. The curve of contact of the tangent cone meets each double line in 3 variable points and 4 fixed points which are pinch points. It meets the 6-fold line in 3 variable points and 18 pinch points. The order of the tangent cone from an arbitrary point not on the surface is 30. Its genus, which is that of the Jacobian of the corresponding net of basic curves, is 49. Hence it has 105 stationary edges and 252 double edges.

If we now apply a quadratic transformation whose fundamental system is a general point O on one of the double lines, say d_5, and the degenerate conic consisting of the 6-fold line k and any other double line, say d_6, the resulting surface is $\sum_8$, of order 8. For it loses the plane kO, or kd_5, six times, the plane d_6O twice, and the plane kd_6 twice. The transform of a general plane of the pencil k is a plane of the pencil d_6, and the relation between these planes is a collineation. Therefore the planes through d_6 meet the new surface in cubics, and d_6 is a 5-fold line on $\sum_8$. The line k is a simple line on $\sum_8$ corresponding to l_5, the residual intersection of the plane kd_5 with $\sum_9$. A plane through O is invariant as a whole. Its points are transformed quadratically. A line meeting d_6 and d_5 meets $\sum_9$ five more times. Since it lies in a plane through O and in a plane of the pencil d_6, its transform is a line passing through the intersection on k and d_5 and meeting $\sum_8$ in 5 more points. Hence the point kd_5 is a triple point of $\sum_8$. The section of $\sum_8$ by the plane kd_6 consists of the 5-fold line d_6, the simple line k, and 2 simple lines e_1 and e_2, which pass through the triple point and correspond to the tangent planes to $\sum_9$ at O. The line l_0 on $\sum_9$ meets d_6 and d_5, and its transform is therefore a line l_0' through the triple point. A general quadric surface intersects $\sum_9$ in a curve whose image is of order 24 and has 8-fold points at $A_1, \cdots, A_8$, and double points at the 6 points B and at A_0. Hence the plane system of $\sum_8$ is the web of duodecimics that have quadruple points at $A_1, \cdots, A_8$, simple points at $B_1, \cdots, B_5$ and at A_0, and pass also through E_1 and E_2, the two points on c_3' (the cubic through B_5) that correspond to O. The plane d_6O meets $\sum_9$ in a septimic having a double point at O. The image of that section is a nonic having triple points at $A_1, \cdots, A_8$ and passing through the 5 points $B_1, \cdots, B_5$, and the points E_1 and E_2. Hence this same nonic is the image of the 5-fold line of $\sum_8$. To a plane section through the 5-fold line corresponds a cubic of the pencil $A_1, \cdots, A_8$. Twelve of these have a node. The image of the triple point of $\sum_8$ is the cubic $c_3{}^5$ containing the points B_5, E_1, E_2, and A_0. To these points correspond the nodal lines k, e_1, e_2, and l_0'. The first 3 lie in the plane kd_6. To the remaining double lines of $\sum_9$ correspond 4 double lines of $\sum_8$, d_1', d_2', d_3', d_4', meeting the 5-fold line. Their images are as before the cubics c_3^1, c_3^2, c_3^3, c_3^4 determined by $B_1, \cdots, B_4$. To plane sections of $\sum_8$ through the triple point correspond the net of nonics having triple points at $A_1, \cdots, A_8$ and simple points at $B_1, \cdots, B_4$. To the pencil of plane sections through a double line, say d_1', correspond the pencil of nonics having triple points at $A_1, \cdots, A_8$, and passing through $B_2, B_3, B_4, B_5, E_1, E_2$. To the section containing d_1' and the node corresponds the sextic having double points at $A_1, \cdots, A_8$ and passing through $B_2B_3B_4$. This section contains l_0'. There is thus a difference between $\sum_8$ and $\sum_9$ in the number of plane sections through a double line that have a node. In both cases to such a pencil of plane sections correspond a pencil of nonics having 8 fixed triple points. But for $\sum_9$ the remaining base points of the pencil are of general position, and the number of curves of the pencil that have an extra node is 32. For $\sum_8$ three base points are on a cubic through the 8 triple points. One nonic of the pencil is composed of this cubic and a sextic. It is in fact the image of the section through the double line and the triple point. The two points, apart from the base points, where the sextic meets the cubic are to be deducted, and the number is 30. It is easy to see that in the next case, $r=5$, this difference is 3, and that it does not increase further with r. For the degenerate curve consists of a cubic, image of the triple point, and a curve of order $3r-6$, which has $(r-2)$-fold points at $A_1, \cdots, A_8$ and an $(r-5)$-fold point at A_0, and hence meets the cubic in 3 other points. The class of $\sum_8$ is 48, or 3 less than the class of $\sum_9$. The Jacobian of a net contains the factor $c_3{}^5$, the image of the node. Removing it, we find that the curve of contact of the tangent cone from an arbitrary point not on the surface, meets each of the 4 double lines in 3 variable points and 4 pinch points, and meets the 5-fold line in 3 variable points and 16 pinch points, the last number being 2 less than for the 6-fold line of $\sum_9$. The Jacobian meets the image of the node in 6 variable points; and hence the tangent cone has a 6-fold edge. The order of the tangent cone, its genus, and the number of its stationary edges are less by 2, 7, and 15 respectively than in $\sum_9$. These differences are all constant in passing from $\sum_{3r-3}$ to $\sum_{3r-4}$. The difference in the number of double edges is a function of r.

The stationary tangent planes to the cubic cone at the triple point of $\sum_8$ are of some interest. The sections of the surface by these planes correspond to nonics of the above mentioned net that have 3-point contact with the cubic $c_3{}^5$. Such a point of contact is a possible 9th triple point for nonics having triple points at $A_1, \cdots, A_8$. A curve of order $3r$ having 8 fixed r-fold points can not have a 9th r-fold point assigned arbitrarily. Halphen* has shown that for $r=3$ the locus of the 9th triple point is a curve that meets any cubic of the pencil in 8 points aside from the base points. A nonic having triple points at $A_1, \cdots, A_8$ and touching the cubic at one of these points will have 3-point contact there with the cubic. The 9th stationary tangent plane is accounted for by A . A nonic having the points $A_1, \cdots, A_8$ for triple points and passing through A_0 meets a cubic of the pencil in 2 more points, either of which corresponds to the other in the involution I_{17} determined by the sextics that have double points at $A_1, \cdots, A_8$. Hence such a nonic tangent to a cubic of the pencil at A_0 must have 3-point

* Bulletin de la Société Mathématique de France, vol. 10 (1881), p. 162; or *Oeuvres*, vol. 2, p. 547.

contact there. This insures the reality of one stationary tangent plane.*

We have seen that in either $\sum_8$ or $\sum_9$ a plane through a double line, or through the 6-fold or the 5-fold line, meets the surface in a curve having 3 variable intersections with the multiple line. When 2 such points coincide the plane of the section is a stationary plane in the developable of the stationary tangent planes of the surface; and the point is a point of contact of the parabolic curve with the multiple line. For example, the images of the sections of $\sum_9$ by planes through the 6-fold line are the cubics of the pencil $A_1, \cdots, A_8$. Let ϕ and ψ be two of them. To find how many cubics of the pencil are tangent to the image of the 6-fold line, that is, the nonic c_9 having triple points at $A_1, \cdots, A_8$ and passing through the 6 points B, we have merely to find the number of intersections of the Jacobian of ϕ, ψ, and c_9 not accounted for at the A's. This Jacobian is of order 12 and has a triple point at each of the points $A_1, \cdots, A_8$ whose 3 tangents coincide with those of c_9.† Hence there are 12 cubics of the pencil tangent to c_9, and therefore 12 such points on the 6-fold line. Similarly there are 6 such points on each double line. Also on $\sum_8$ there are 12 such points on the 5-fold line and 6 on each double line. The genus of a curve of order n on $\sum_9$ has the upper limit $(4n^2+12n+135)/72$. On $\sum_8$ this limit is $(n^2+4n+34)/16$.

The system of rational quartics on the two surfaces deserves notice. To the sextics of the web having double points at $A_1, \cdots, A_8$ correspond octavic curves. Such a sextic may be composed of the line joining 2 of the points A and the quintic having double points at the other 6 and passing through the first 2. The line and the quintic intersect in 3 more points invariant under I_{17}. We get thus on either surface 2 rational quartics intersecting in 3 points, and on $\sum_8$ passing each once through the node. There are 28 such pairs. Similarly the sextic may be composed of the conic through 5 of the points A and the quartic having double points at the other 3 and passing through the first 5. This gives 56 pairs of similar quartics. The quartics corresponding to the points $A_1, \cdots, A_8$ are of this type. With the quartic corresponding to A_i is paired the

* See Snyder, American Journal of Mathematics, vol. 33 (1911), p. 328.

† Hilton, *Higher Plane Curves*, p. 110.

quartic (also rational) whose image is the sextic having a triple point at A_i and double points at the other 7. This sextic corresponds to A_i in I_{17}. There are thus 92 pairs of such quartics on both surfaces.

The extension of the above is obvious. For the sake of completeness a summary of the general case is added. The following applies to $r\geqq 4$. There is a web of curves of order $3r$, and of genus $3r-5$, which have in common 8 r-fold points $A_1, \cdots, A_8$, $3(r-2)$ simple points B, and an $(r-3)$-fold point at A_0. Two curves of the web have $3(r-1)$ variable intersections. There is one curve c_{3r-3} which has $(r-1)$-fold points at $A_1, \cdots, A_8$, passes through the $3(r-2)$ points B, and has an $(r-4)$-fold point at A_0. We have, therefore, a rational surface $\sum_{3r-3}$, whose plane sections correspond to the curves of the web. The double curve of this surface consists of a $(3r-6)$-fold line and $3r-6$ double lines, which meet the former, but are skew to each other. The surface has also $3r-6$ simple lines and a rational curve of order $r-3$, which meets each double line once and the $(3r-6)$-fold line $r-4$ times. The image of the $(3r-6)$-fold line is c_{3r-3}. The images of the double lines are the cubics of the pencil $A_1, \cdots, A_8$ determined respectively by the points B. The class of $\sum_{3r-3}$ is $3(6r-7)$. The curve of contact of the tangent cone from an arbitrary point not on the surface, meets each double line in 3 variable points and 4 pinch points. It meets the $(3r-6)$-fold line in 3 variable points and $6(2r-5)$ pinch points. The order of the tangent cone is $6(2r-3)$. Its genus is $24r-47$; and it has $3(18r-37)$ stationary edges, and $12(6r^2-26r+29)$ double edges.

Applying a quadratic transformation whose fixed conic is the $(3r-6)$-fold line and a double line d_i, and whose fixed point O is on another double line d_j, we obtain a surface $\sum_{3r-4}$ of order one lower, which has a triple point at the intersection of d_j with the $(3r-6)$-fold line, and whose double curve is a $(3r-7)$-fold line, coinciding with d_i, and $3r-8$ double lines meeting the former, but skew to each other. There are $3r-8$ simple lines and a rational curve of order $r-3$ which passes once through the triple point and meets each double line once and the $(3r-7)$-fold line $r-4$ times. The class of the new surface is less by 3. As in $\sum_{3r-3}$, the curve of contact of the tangent cone of $\sum_{3r-4}$ has 3 variable intersections with each double line and with the $(3r-7)$-fold line, and there are, as before, 4 pinch points on each double line. But the number of pinch points on the $(3r-7)$-fold line is 2 less. The curve of contact of the tangent cone passes 6 times through the triple point. As remarked above, the order of the tangent cone, its genus, and the number of its stationary edges are less by 2, 7, and 15 respectively than in $\sum_{3r-3}$. The number of double edges in $\sum_{3r-4}$ is $3(24r^2-112r+137)$, a reduction of $24r-63$. The plane system of $\sum_{3r-4}$ is the same as that of $\sum_{3r-3}$ with the exception that B_i is dropped, and two associated points on the cubic which is the image of d_j are added. The image of the $(3r-7)$-fold line is the curve c'_{3r-3} which has $(r-1)$-fold points at $A_1, \cdots, A_8$, passes through the B's (except B_i) and through the 2 associated points just mentioned, and has an $(r-4)$-fold point at A_0. There are $6(r-2)$ plane sections of either surface containing the $(3r-6)$-fold or the $(3r-7)$-fold line and tangent to it. Through a double line on either surface are 6 sections that are tangent to it. The planes of these sections are doubly stationary, that is, stationary planes in the developable of the stationary tangent planes of the surface. The genus of a curve of order n on $\sum_{3r-3}$ does not exceed the greatest integer in

$$\frac{4n^2+12n(r-3)+3\,(6r^2-19r+25)}{24(r-1)}.$$

The corresponding expression for $\sum_{3r-4}$ is

$$\frac{4n^2+4n(3r-8)+18r^2-51r+52}{8\,(3r-4)}.$$

Exactly as in $\sum_9$ and $\sum_8$ there are on $\sum_{3r-3}$ and $\sum_{3r-4}$ 92 pairs of rational curves of order r. The two curves of such a pair meet 3 times on the curve whose image is the locus of invariant points in I_{17}, and in $\sum_{3r-4}$ both pass once through the triple point. To the locus of invariant points in I_{17} corresponds on either surface a curve of order $3r$, which in $\sum_{3r-4}$ passes 3 times through the triple point.

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