ON LOCI OF \((r-2)\)-SPACES INCIDENT WITH CURVES IN \(r\)-SPACE

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Let \(s_t\) lines and \(t\) curves \(C^{m_1}, C^{m_2}, \ldots, C^{m_t}\) of orders \(m_1, m_2, \ldots, m_t\) and deficiencies \(p_1, p_2, \ldots, p_t\), respectively, be given in general positions in \(r\)-space. In this paper, we propose to determine the number, \(N_r^{(i)}\), of \((r-2)\)-spaces that are incident with the \(s_t\) lines and meet \(C^{m_1}\) \(n_1\) times, \(C^{m_2}\) \(n_2\) times, \(\ldots, C^{m_t}\) \(n_t\) times, where

\[
(A) \quad s_t + n_1 + n_2 + \cdots + n_t = 2r - 2,
\]

and to deduce a few consequences from the formula for this number. The formula which we shall derive is obviously a function of \(r, n_i, m_i\), and \(p_i, (i = 1, 2, \ldots, t)\). The derivation of this formula can be accomplished algebraically,* or by Schubert's symbolic calculus,† by the functional method,‡ or by the method of decomposition.‡ In the present work we find it convenient to adopt the method last named as it yields the desired result with the least difficulty.

A curve \(C^m\) may be decomposed in various ways into component curves the sum of whose orders is equal to \(m\), but we wish to decompose it completely, that is, into \(m\) lines forming a skew polygon \(\Gamma\) of \(m\) sides and \(Q^{(1)} = m - 1 + p\) vertices where \(p\) is the deficiency of \(C^m\). The non-adjacent vertices of \(\Gamma\) arrange themselves in groups each consisting of a certain number, \(q\), of members. Let \(Q^{(q)}\) denote the number of such groups. As we shall have frequent use for this number, we record the following which can be easily verified:

\[
Q^{(1)} = \binom{m - 1}{1} + \binom{p}{1},
\]

† Schubert, Kalkül der Abzählenden Geometrie, Leipzig, 1879.
‡ Severi, Riflessioni intorno ai problemi numerativi concernenti le curve algebriche, Rendiconti Istituto Lombardo, (2), vol. 54 (1921), pp. 243–254.
\[ Q^{(2)} = \binom{m-2}{2} + \binom{m-3}{2} + \binom{0}{2}, \]

\[ Q^{(q)} = \sum_{j=0}^{q} \binom{m-q-j}{q-j} \binom{0}{j}; \]

\( Q^{(0)} \) is to be taken equal to unity.

Now let \( t = 0 \). Then the symbol \( N_r^{(0)} \) denotes the number of \( (r-2) \)-spaces incident with \( s_0 = 2r-2 \) general lines in \( r \)-space. Since the number of lines incident with \( 2r-2 \) general \( (r-2) \)-spaces given in \( r \)-space is

\[ \frac{(2r-2)!}{r!(r-1)!}, \]

we assume by the principle of duality, or we can prove independently, that \( N_r^{(0)} \) is equal to this number; that is,

\[ N_r^{(0)} = \frac{(2r-2)!}{r!(r-1)!}. \]

Diminishing \( r \) by \( w \), we have

\[ N_{r-w}^{(0)} = \frac{(2r-2w-2)!}{(r-w)!(r-w-1)!}, \]

for the number of \( (r-w-2) \)-spaces that meet \( 2r-2w-2 \) general lines in \( (r-w) \)-space, which is also the number of \( (r-2) \)-spaces that pass through \( w \) general points and meet \( 2r-2w-2 \) general lines in \( r \)-space.

We proceed now to determine, for the case \( t = 1 \), the number \( N_r^{(1)} \) of \( (r-2) \)-spaces that meet \( n^1 \) times a given curve \( C^{m_1} \) and are incident with \( s_1 = 2r-2-n_1 \) given lines in \( r \)-space. Replace \( C^{m_1} \) by an \( m_1 \)-sided skew polygon \( \Gamma \) with \( Q_1^{(1)} = m_1-1+\rho_1 \) vertices. Any \( m_1 \) general lines determine \( n_1 \) by \( n_1 \) with the \( s_1 \) given lines

\[ \binom{m_1}{n_1} N_r^{(0)} \]

(\(r-2\))-spaces all incident with the \(s_1\) given lines and each incident with \(n_1\) of the \(m_1\) lines. But the \(m_1\) lines or sides of \(\Gamma_1\) have \(Q_1^{(1)}\) incidences each on two of the lines. Through each vertex of \(\Gamma_1\) pass

\[
\binom{m_1-2}{n_1-2} N_r^{(0)}
\]

(\(r-2\))-spaces all incident with the \(s_1\) given lines and each incident with \(n_1-2\) of the \(m_1-2\) sides on which the vertex does not lie. Therefore the \(Q_1^{(1)}\) vertices of \(\Gamma_1\) determine

\[
\binom{m_1-2}{n_1-2} N_r^{(0)} Q_1^{(1)}
\]

such (\(r-2\))-spaces. As these (\(r-2\))-spaces are improper \(n_1\)-uple secant (\(r-2\))-spaces of the degenerate curve \(C_{m_1}\) incident with the \(s_1\) lines, we deduct their number from

\[
\binom{m_1}{n_2} N_r^{(0)}
\]

To the result we now add

\[
\binom{m_1-4}{n_2-4} N_r^{(0)} Q_1^{(1)}
\]

which is the number of (\(r-2\))-spaces each passing through a pair of non-adjacent vertices of \(\Gamma_1\) and meeting the \(s_1\) given lines and \(n_1-4\) of the \(m_1-4\) sides of \(\Gamma_1\) not passing through the vertices. Continuing in this manner, we find

\[
N_r^{(1)} = \sum_{q_i=0}^{h_1} (-1)^{q_i} \binom{m_1 - 2q_1}{n_1 - 2q_1} N_r^{(0)} Q_1^{(q_1)}
\]

where \(h_1 = n_1/2\) if \(n_1\) is even and \(h_1 = (n_1-1)/2\) if \(n_1\) is odd. Replacing \(r\) by \(r-w\), we have

\[
N_{r-w}^{(1)} = \sum_{q_i=0}^{h_1} (-1)^{q_i} \binom{m_1 - 2q_1}{n_1 - 2q_1} N_{r-w}^{(0)} Q_1^{(q_1)}
\]

as the number of (\(r-2\))-spaces that pass through \(w\) given general points and meet a given curve \(C_{m_1} n_1\) times and also meet \(s_1-2w = 2r-2w-2-n_1\) given lines in \(r\)-space.

Now let \(t = 2\). Then \(s_2 = 2r-2-n_1-n_2\). To determine the number, \(N_r^{(2)}\), of (\(r-2\))-spaces that meet two curves \(C_{m_1}\), \(C_{m_2}\) respectively \(n_1\), \(n_2\) times and are incident with \(s_2\) given gen-
eral lines, we decompose $C_m$ into $m$ lines forming an $m$-sided skew polygon $\Gamma_2$ with $Q_1 = m - 1 + p_2$ vertices. There are
\[
\binom{m_2}{n_2} N_r^{(1)}
\]
$(r-2)$-spaces incident with any $m$ general lines of $r$-spaces $n_2$ at a time which are also incident with the $s_2$ given lines. From this number we deduct
\[
\binom{m_2 - 2}{n_2 - 2} N_{r-1} Q_2^{(1)},
\]
which is the number of $(r-2)$-spaces all incident with the $s_2$ given lines and each incident with a vertex of $\Gamma_2$ and with $n_2 - 2$ of the $m_2 - 2$ sides of $\Gamma_2$ on which the vertex does not lie. Continuing as in the preceding paragraph, we find
\[
N_r^{(2)} = \sum_{q=0}^{h_2} (-1)^q \binom{m_2 - 2q_2}{n_2 - 2q_2} N_{r-2q_2} Q_2^{(q_2)},
\]
where $h_2 = n_2/2$ if $n_2$ is even and $h_2 = (n_2 - 1)/2$ if $n_2$ is odd. Putting $w = q_2$ in $(2a)$ and substituting the result in the above we have, after simplifying,
\[
N_r^{(3)} = \sum_{q_1=0}^{h_1} \sum_{q_2=0}^{h_2} (-1)^{q_1+q_2} \binom{m_1 - 2q_1}{n_1 - 2q_1} \binom{m_2 - 2q_2}{n_2 - 2q_2} N_{r-2q_2} Q_2^{(q_2)} Q_1^{(q_1)}.
\]
To determine $N_r^{(3)}, N_r^{(4)}, \ldots$, we proceed in a similar manner. Finally, we arrive at the desired formula:
\[
N_r^{(4)} = \sum_{q_1=0}^{h_1} \sum_{q_2=0}^{h_2} \ldots \sum_{q_t=0}^{h_t} (-1)^{q_t} N_{r-q_t}^{(t)}
\]
\[
\times \left( \binom{m_1 - 2q_1}{n_1 - 2q_1} \right) \left( \binom{m_2 - 2q_2}{n_2 - 2q_2} \right) \ldots \left( \binom{m_t - 2q_t}{n_t - 2q_t} \right) Q_1^{(q_1)} Q_2^{(q_2)} \ldots Q_t^{(q_t)},
\]
where
\[
q = q_1 + q_2 + \ldots + q_t,
\]
and
\[
h_i = n_i/2, \text{ if } n_i \text{ is even},
\]
and
\[
h_i = (n_i - 1)/2, \text{ if } n_i \text{ is odd}.
\]
Now we deduce a few consequences from this formula. For
$t=0$ and $t=1$, we have (1) and (2) respectively. If we put in (2) $n_1 = 2r-2$, we obtain, since $h_1 = r-1$,

$$N_r^{(1)} = \sum_{q_1=0}^{r-1} (-1)^{q_1} \binom{m_1 - 2q_1}{2r - 2 - 2q_1} N_{r-q_1}^{(0)} Q_1^{(q_1)},$$

as the number of $(2r-2)$-secant $(r-2)$-spaces of an $r$-space curve $C_{m_1}$. For $r=2$ and $r=3$, (5) becomes respectively

$$N_2^{(1)} = \binom{m_1}{2} - Q_1^{(1)} = \frac{1}{2} (m_1 - 1)(m_1 - 2) - \rho_1,$$

and

$$N_3^{(1)} = \sum_{q_1=0}^{2} (-1)^{q_1} \binom{m_1 - 2q_1}{4 - 2q_1} N_{3-q_1}^{(0)} Q_1^{(q_1)} = \frac{1}{12} (m_1 - 2)(m_1 - 3)^2(m_1 - 4) - \frac{1}{2} (m_1 - 3)(m_1 - 4) \rho_1 + \frac{1}{2} \rho_1 (\rho_1 - 1),$$

the former giving the number of double points on a plane curve $C_{m_1}$ of deficiency $\rho_1$ and the latter giving the number of quadri-secant lines of a 3-space curve $C_{m_1}$ of deficiency $\rho_1$.

If we put $m_1 = 2r-1$, $\rho_1 = 0$ in (5), we have, taking account of (B) and (1a),

$$\sum_{q_1=0}^{r-1} (-1)^{q_1} \frac{(2r - q_1 - 1)!}{q_1!(r - q_1)!(r - q_1 - 1)!},$$

which is equal to unity. That is, a rational curve $C^{2r-1}$ of order $2r-1$ in $r$-space has one and only one $(2r-2)$-secant $(r-2)$-space.

Again, if we put in (2) $n_1 = 2r-3$ and hence $s_1 = 1$, $h_1 = r-2$, we obtain

$$N_r^{(1)} = \sum_{q_1=0}^{r-2} (-1)^{q_1} \binom{m_1 - 2q_1}{2r - 3 - 2q_1} N_{r-q_1}^{(0)} Q_1^{(q_1)}.$$

This is the number of $(2r-3)$-secant $(r-2)$-spaces of an $r$-space curve $C_{m_1}$ that meet a given line, and is therefore the order of the hypersurface formed by the $\infty^1 (2r-2)$-secant $(r-2)$-spaces of $C_{m_1}$. For $r=2$, the formula gives $m_1$, that is, the locus of points on a plane curve $C_{m_1}$ is the curve itself. For $r=3$, we have
\[
\overline{N}_3^{(1)} = \left( \frac{m_1}{3} \right) - \left( \frac{m_1 - 2}{1} \right) (m_1 - 1 + p_1)
\]
\[
= \frac{1}{3} (m_1 - 1)(m_1 - 2)(m_1 - 3) - (m_1 - 2)p_1,
\]
for the order of the trisecant surface of a 3-space curve \( C^{m_1} \).

It is of interest to note that the result of substituting \( m_1 = 2r - 2 \) and \( p_1 = 0 \) in (6) is, if account be taken of (B) and (1a),
\[
\sum_{q_1=0}^{r-2} (-1)^{q_1} \frac{(2r - q_1 - 2)!(2r - 2q_1 - 2)}{q_1!(r - q_1)!(r - q_1 - 1)!} = 2.
\]
Therefore, the locus of the \( \infty^1 \) \((r-2)\)-spaces that meet a rational \( r \)-space curve \( C^{2r-2} \) of order \( 2r-2 \) is always a quadric hypersurface.

Returning to the general formula (4), we see that it is identical with (3) if \( t = 2 \). Let \( s_2 = 0 \). Then, from (A), \( n_1 + n_2 = 2r - 2 \). Consider the case \( n_1 = n_2 = r - 1 \). Then (3) becomes
\[
(7) \quad N_r^{(2)} = \sum_{q_1=0}^{h_1} \sum_{q_2=0}^{h_2} (-1)^{q_1+q_2} \left( \frac{m_1 - 2q_1}{r - 1 - 2q_1} \right) \left( \frac{m_2 - 2q_2}{r - 1 - 2q_2} \right) N_{r-q_1-q_2} Q_1^{(q_1)} Q_2^{(q_2)},
\]
where \( h_1 = h_2 = (r-1)/2 \) if \( r \) is odd and \( h_1 = h_2 = (r-2)/2 \) if \( r \) is even. This gives the number of common \((r - 1)\)-secant \((r - 2)\) spaces of two curves \( C^{m_1} \) and \( C^{m_2} \) in \( r \)-space. The case \( m_1 = m_2 = r \) and \( p_1 = p_2 = 0 \) is worth noting. Formula (7) for this case gives
\[
\sum_{q_1=0}^{h_1} \sum_{q_2=0}^{h_2} (-1)^{q_1+q_2} \frac{(r - 2q_1)(r - 2q_2)}{(r - q_1 - q_2)}
\]
\[
\times \left( \frac{r - q_1}{q_1} \right) \left( \frac{r - q_2}{q_2} \right) \left( \frac{2r - 2q_1 - 2q_2 - 2}{r - q_1 - q_2 - 1} \right)
\]
\[
= \sum_{j=0}^{k} (r - 2j)^2,
\]
\[
= \frac{1}{6} r(r + 1)(r + 2)
\]

\[\begin{cases} k = r/2 & \text{if } r \text{ is even} \\ k = (r - 1)/2 & \text{if } r \text{ is odd} \end{cases}\]
as the number of common \((r-1)\)-secant \((r-2)\)-spaces of two normal curves of order \(r\) in \(r\)-space.\(^*\) Thus, two twisted cubic curves in 3-space have 10 common secant lines.

As another application of the general formula (4) we give the following. Let

\[ n_1 + n_2 + \cdots + n_t = 2r - 4. \]

Hence, from (A), \(s_t = 2\). Then formula (4) gives the number of \((r-2)\)-spaces that are incident with two given lines and meet \(t\) curves \(C^{ni} n_i\) times where \(\sum_{i=1}^{t} n_i = 2r - 4\). This is also the order of the hypersurface \(V_{r_t}\) formed by the \(\infty 1\) \((r-2)\)-spaces that are incident with a given line and meet \(C^{ni} n_i\) times. The \(\infty 2\) \((r-2)\)-spaces incident with \(C^{ni} n_i\) times meet a general 3-space, and in particular, a 3-space passing through \(l\), in the lines of a congruence \(K\) the sum of whose order \(\mu\) and class \(\nu\) is the order of the hypersurface \(V_{r_t}\). The order of \(K\) is evidently \(N_{r_t}\), obtained from (4) by changing \(r\) to \(r-1\), for this is the number of \((r-2)\)-spaces that pass through a given point and meet \(C^{ni} n_i\) times. Therefore, the class of \(K\) or the number of \((r-2)\)-spaces that meet \(C^{ni} n_i\) times and meet a given plane in lines is

\[ \nu = N_{r_t}^{(t)} - N_{r_{t-1}}^{(t)}. \]

\(^*\) This result can also be obtained from (5) by putting \(m_1 = 2r\) and \(p_1 = -1\).