A NOTE CONCERNING CACTOIDS*

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A cactoid‡ $M$ is a bounded continuous curve lying in space of three dimensions and such that (a) every maximal cyclic curve§ of $M$ is a simple closed surface and (b) no point of $M$ lies in a bounded complementary domain of any subcontinuum of $M$. There exists a bounded acyclic|| continuous curve $C$ such that every bounded acyclic continuous curve is homeomorphic with a subset of $C$. Now Whyburn has shown¶ that with respect to its cyclic elements every continuous curve is acyclic. Moreover the cyclic elements of a cactoid are either points or topological spheres. Thus this question naturally arises: Does there exist a cactoid $C$ such that every cactoid is homeomorphic with a subset of $C$? The object of the present paper is to answer this question negatively.

THEOREM 1. There does not exist a cactoid $C$ such that every cactoid is homeomorphic with a subset of $C$.

PROOF. Let $g$ be any infinite set of distinct positive integers $d_1, d_2, d_3, \ldots$. Let $K$ denote a non-dense perfect point set on the interval $0 \leq x \leq 1$ containing the end points of this interval. The complementary segments of $K$ can be labeled

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§ For a definition of this term, and of the term cyclic element, see G. T. Whyburn, Concerning the structure of a continuous curve, American Journal of Mathematics, vol. 50 (1928), p. 167.
s_{ij}(i, j = 1, 2, 3, \cdots) in such a manner that for every i' and
two distinct points U and V of K there is a j such that
the segment s_{ij} is between U and V. For each i and j there
exists a continuum \( M_{ij} \) which is the sum of \( d_i \) spheres
\( A_1, A_2, \cdots, A_{d_i} \), where a diameter of \( A_k (k \leq d_i) \) is a subset of
the interval \( s_{ij} \), \( A_k \) and \( A_{k+1} \) are tangent externally, and \( A_1 \) and
\( A_{d_i} \) respectively contain the end points of \( s_{ij} \). Let \( G_\varphi \) denote the
collection whose elements are the continua \( M_{ij}(i, j = 1, 2, 3, \cdots) \)
and those points of K which do not belong to any continuum
\( M_{ij} \). Then \( G_\varphi \) is an upper semi-continuous collection, and is
an arc with respect to its elements. Moreover, for each i the
elements of \( G_\varphi \) which are the sum of \( d_i \) spheres form a set which
is everywhere dense on this arc. Let \( C_\varphi^* \) be the point set
\( K + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} M_{ij} \). Then \( C_\varphi^* \) is a cactoid. Let \( P \) and \( Q \) denote
the end points of the interval \( 0 \leq x \leq 1 \). Then it is easy to see
that if D is any point of \( C_\varphi^* \) other that P and Q, there exist
in \( C_\varphi^* \) arcs PD and QD which have only D in common. Ob­
viously also if \( g_1 \) and \( g_2 \) are two infinite sets of distinct positive
integers and \( g_1 \) contains an integer not in \( g_2 \) then \( C_\varphi^* \) and \( C_\varphi^* \)
are not homeomorphic. Now there exists an uncountable
collection \( (g) \) such that each element of \( (g) \) is an infinite set of
distinct positive integers, and for each two elements \( g_1 \) and \( g_2 \)
of \( (g) \) there is an integer which belongs to one of them but not
to the other.

Suppose \( C \) is a cactoid such that every cactoid is homeo­
morphic with a subset of \( C \). Then for each element g of \( (g) \) the
set \( C \) contains a cactoid \( C_\varphi \) which is homeomorphic with the
cactoid \( C_\varphi^* \) defined above. Let \( P_\varphi \) and \( Q_\varphi \) be the points of \( C_\varphi \)
which correspond to the points P and Q under a transformation
throwing \( C_\varphi^* \) into \( C_\varphi \). As \( (g) \) is uncountable it is easy to see
that there exists an infinite sequence \( g_1, g_2, g_3, \cdots \), of elements
of \( (g) \) such that \( P_{g_1} \) and \( Q_{g_1} \), respectively, are sequential limit
points of the sequences \( P_{g_2}, P_{g_3}, P_{g_4}, \cdots \), and \( Q_{g_2}, Q_{g_3},
Q_{g_4}, \cdots \). As \( C \) is a continuous curve and \( P_{g_1} \neq Q_{g_1} \), there exists
an \( n(n > 1) \) such that \( C \) contains arcs \( P_{g_1}P_{g_n} \) and \( Q_{g_1}Q_{g_n} \) which
have no points in common. Suppose \( C_{g_n} \) contains a point D
(different from \( P_{g_n} \) and \( Q_{g_n} \)) which does not belong to \( C_{g_1} \). Now
\( C_{g_n} \) contains arcs \( P_{g_n}D \) and \( Q_{g_n}D \) having only D in common.
Hence there exists an arc \( XDY \) in \( C \) with only X and Y in
\( C_{g_1} \). There exists an arc \( XBY \) which is a subset of \( C_{g_1} \). As the
maximal cyclic curves of $C$ are spheres it follows that the simple closed curve $XYDBX$ is a subset of a sphere $S$ which belongs to $C$. Now the arc $XBY$ contains a subarc which is a subset of a sphere $T$ belonging to $C_{01}$. Then $S$ and $T$ have more than one point in common, and hence are identical. Then $C_{01}$ contains $D$, contrary to supposition, whence $C_{0n}$ is a subset of $C_{01}$. Likewise $C_{01}$ is a subset of $C_{0n}$. As this is impossible we see that the above supposition has led to a contradiction and the theorem is proved.

In glancing over the proof one can see that the only property used of the topological sphere (which is not also a property of every compact, cyclicly connected continuous curve) is that it is not homeomorphic with a proper subset of itself. Thus the proof suffices for the following theorem.

Theorem 2. If $M$ is a class of compact continuous curves whose maximal cyclic curves are homeomorphic but no one is homeomorphic with a proper subset of itself, then there is no universal curve of class $M$; that is, no curve $C$ of class $M$ such that every curve of class $M$ is homeomorphic with a subset of $C$.