If two pairs of operators $p$ and $\tau$ and $\Sigma$ and $\tau$ which generate the group are so selected that one of them $p$ and $\tau$ satisfy a particular one of the sets of conditions imposed on $S$ and $T$ and if $p$ happens to be the inverse of $\Sigma$ then $\Sigma$ and $\tau$ satisfy identically the same relations as $p$ and $\tau$. This follows from the fact that there is an outer isomorphism which transforms $p$ into its inverse and leaves $T$ invariant.

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ONE-PARAMETER LINEAR FUNCTIONAL GROUPS IN SEVERAL FUNCTIONS OF TWO VARIABLES*

BY A. D. MICHAL

Let

\begin{equation}
\frac{\partial z(x, y; \tau)}{\partial \tau} = iH_{ij}(x, y; \tau), \quad (i, j = 1, 2, \cdots, n),
\end{equation}

be a given set of $n^2$ real continuous functions of the two real variables $x$ and $y$ defined over the triangular region $T: a \leq x \leq y \leq b$. Consider the system of $n$ integro-differential equations

\begin{equation}
\frac{\partial z(x, y; \tau)}{\partial \tau} = \int_0^y \int H(x, s)z(s, y; \tau)ds.
\end{equation}

We assume that the reader is conversant with the theory of permutable functions and functions of composition, a subject initiated by the illustrious Vito Volterra.† Griffith C. Evans‡

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* Presented to the Society, December 30, 1930.
† Throughout the whole paper we shall adhere to the convention of letting a repetition of an index in a term once as a subscript and once as a superscript denote summation with respect to that index over the values $1, 2, \cdots, n$.
‡ See, for example, Volterra et Pérès, Leçons sur la Composition et les Fonctions Permutables.
§ See his Cambridge Colloquium Lectures, Functionals and their Applications.
has given some very noteworthy contributions to this subject.

Define a set of \( n^2 \) functions \( iK(x, y; \tau) \) in terms of the given set of functions \( iH(x, y) \) by

\[
(4) \quad iK(x, y; \tau) = iH(x, y) \tau + \frac{k!}{2!}iH^{(2)}(x, y) \tau^2 + \ldots.
\]

For \( n = 1 \), this set reduces to the well known Volterra transcendental.

**Theorem 1.** Under the above hypotheses on the given functions \( iH(x, y) \), the system of integro-differential equations (2) possesses one and only one solution \( \varphi(x, y; \tau) \) continuous in \( \tau \), \( \varphi(x, y; \tau) \) analytic in \( \tau \) (in fact entire in \( \tau \)), and such that it takes on the initial conditions

\[
(5) \quad \varphi(x, y; 0) = \varphi(x, y),
\]

\( (\varphi(x, y) \) continuous in \( T) \) for \( \tau = 0. \) This unique solution can be written in the form

\[
(6) \quad \varphi(x, y; \tau) = \varphi(x, y) + \sum_{k=1}^{n^2} iK(\tau; \tau) \varphi(x, y),
\]

where the \( n^2 \) functions \( iK \) are the generalized Volterra transcendentals (4).

The proof of this theorem is obtained by throwing the integro-differential system (2) into the form of a system of non-linear integral equations

\[
(7) \quad \varphi(x, y; \tau) = \varphi(x, y) + \int_0^\tau iH^{(2)}(x, y; \sigma) d\sigma,
\]

and then solving this system by the method of successive approximations. We are thus led to consider the functions \( \varphi(x, y; \tau) \) given by (6). By calculation one sees that this set of functions is a formal solution of the integro-differential system and takes on the given initial conditions. In fact the transcendentals \( iK(x, y; \tau) \) satisfy the integro-differential equations

\[
(8) \quad \frac{\partial iK(x, y; \tau)}{\partial \tau} = iH(x, y) + \sum_{k=1}^{n^2} iH^{(2)}iK(x, y; \tau),
\]
and hence the $\zeta(x, y; \tau)$ of (6) yield a formal solution of the system (2).

It follows directly from the definition (4) that the generalized Volterra transcendental functions are \textit{entire} functions of $\tau$ and continuous in $x$ and $y$ throughout the region $\mathcal{T}$. Hence the set of functions (6) form an actual solution of (2) of the required sort.

If $\zeta(x, y; \tau)$ is another solution of (2) of the required sort, it follows from (7) that it is possible to find a positive number $A < 1$ such that

$$\max |\zeta(x, y; \tau) - \zeta(x, y; \tau)| \leq A \max |\zeta(x, y; \tau) - \zeta(x, y; \tau)|,$$

for sufficiently small values of $\tau$. But $\zeta$ and $\zeta$ are analytic functions of $\tau$. Hence

$$\zeta(x, y; \tau) \equiv \zeta(x, y; \tau).$$

This shows that the solution of (2) of the required sort is unique, which completes the proof of our theorem.

Thus the integro-differential system (2) generates a continuous one-parameter family of functional transformations (6). Moreover, one can verify that the generalized Volterra transcendental $\zeta$ possess the following integral addition theorem:

$$^{(9)} \zeta(x, y; \tau_1 + \tau_2) = \zeta(x, y; \tau_1)$$

$$+ \zeta(x, y; \tau_2) + \zeta(\cdots; \tau_2) \star \zeta(x, y; \tau_1),$$

and that this integral addition theorem characterizes the $n^2$ transcendental $\zeta$. We have thus arrived at the following fundamental theorem.

**Theorem 2.** The integro-differential equations (2) (infinitesimal transformations) generate the one-parameter continuous group of linear functional transformations (6) in the functions $\zeta(x, y)$, $\zeta(x, y)$, $\cdots$, $\zeta(x, y)$. Moreover the identity transformation is obtained by putting the parameter $\tau$ equal to zero.

Consider the particular case in which

$$^{(10)} \left\{ \begin{array}{l} \zeta(x, y) = 1, \text{ if } j = n - i + 1, \\ \zeta(x, y) = 0, \text{ otherwise.} \end{array} \right.$$

The integro-differential system (2) for this case becomes
\[ \frac{\partial s(x, y; \tau)}{\partial \tau} = 1^{*_{n-i+1}} s(x, y; \tau). \]

By calculation we obtain

\[ jK(x, y; \tau) = \sum_{n=1}^{\infty} \frac{\tau^{2n}(y - x)^{2n-1}}{(2n)!(2n - 1)!}, \quad \text{if } j = i, \]

\[ = \sum_{n=1}^{\infty} \frac{\tau^{2n-1}(y - x)^{2n-2}}{(2n - 1)!(2n - 2)!}, \quad \text{if } j = n - i + 1, \]

\[ = 0, \quad \text{otherwise.} \]

These expressions can be written in terms of the Bessel function \( J_i(z) \) of order one.\(^*\) Putting \( u = y - x \), we obtain

\[ jK(x, y; \tau) = \frac{\tau^{1/2} u^{-1/2}}{2} \left\{ I_1[2(\tau u)^{1/2}] + J_1[-2(\tau u)^{1/2}] \right\}, \quad \text{if } j = i, \]

\[ = \frac{\tau^{1/2} u^{-1/2}}{2} \left\{ I_1[2(\tau u)^{1/2}] - J_1[-2(\tau u)^{1/2}] \right\}, \quad \text{if } j = n - i + 1, \]

\[ = 0, \quad \text{otherwise.} \]

In these formulas,

\[ I_1(z) = -iJ_1(iz), \quad (i = (-1)^{1/2}). \]

For \( j = i \) the integral addition theorems (9) for the special case (13) yield the addition theorem

\[ \text{For } j = i \text{ the integral addition theorems (9) for the special case (13) yield the addition theorem} \]

\[ \begin{align*}
(\tau_1 + \tau_2)^{1/2} u^{-1/2} & \left\{ I_1[(\tau_1 + \tau_2)u] + \tau_1^{1/2} A[\tau_1 u] + \tau_2^{1/2} A[\tau_2 u] \right. \\
& \left. + (\tau_1 \tau_2)^{1/2} \int_0^u [(u - v)v]^{-1/2} \right. \\
& \left. \cdot \{ J_1[-2(\tau_2(u - v))^{1/2}]J_1[-2(\tau_1 v)^{1/2}] \right. \\
& \left. + I_1[2(\tau_2(u - v))^{1/2}]J_1[2(\tau_1 v)^{1/2}] \} dv, \right.
\end{align*} \]

where

\[ A[\tau u] = J_1[-2(\tau u)^{1/2}] + I_1[2(\tau u)^{1/2}]. \]

By an application of a known theorem* one can verify that \( I_1(z) \) possesses the following integral addition theorem:

\[
\left( (\tau_1 + \tau_2)^{1/2} u^{-1/2} \right) I_1 \left[ 2 ((\tau_1 + \tau_2) u)^{1/2} \right] = \tau_1^{1/2} u^{-1/2} I_1 \left[ 2 (\tau_1 u)^{1/2} \right] + \tau_2^{1/2} u^{-1/2} I_1 \left[ 2 (\tau_2 u)^{1/2} \right] + (\tau_1 \tau_2)^{1/2} \int_0^u [(u - v) v]^{-1/2} I_1 \left[ 2 (\tau_2 (u - v))^{1/2} \right] I_1 \left[ 2 (\tau_1 v)^{1/2} \right] dv.
\]

Moreover

\[
J_1(z) = i I_1(-iz), \quad (i = (-1)^{1/2}).
\]

Hence by calculation we see that \( J_1 [-2 (\tau_2 u)^{1/2}] \) has precisely the same integral addition theorem as that of \( I_1 [2 (\tau_2 u)^{1/2}] \).

The remaining relations (9) for the particular system (10) do not yield any essentially new integral addition theorems for the Bessel functions.

**Theorem 3.** The group properties of the continuous one-parameter family of functional transformations that is generated by the integro-differential system (11) are translated by the integral addition theorem

\[
(\tau_1 + \tau_2)^{1/2} u^{-1/2} J_1 [-2 ((\tau_1 + \tau_2) u)^{1/2}] = \tau_1^{1/2} u^{-1/2} J_1 [-2 (\tau_1 u)^{1/2}] + \tau_2^{1/2} u^{-1/2} J_1 [-2 (\tau_2 u)^{1/2}] + (\tau_1 \tau_2)^{1/2} \int_0^u [(u - v) v]^{-1/2} J_1 [-2 (\tau_2 (u - v))^{1/2}] J_1 [-2 (\tau_1 v)^{1/2}] dv
\]

for the Bessel function \( J_1(z) \) of order one.

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