TYPES OF SERIES AND TYPES OF SUMMABILITY*

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1. *Introduction.* In most historical accounts of the theory of divergent series considerable stress is laid on the dictum of Abel and Cauchy with regard to their use and its influence on the study of such series. While not wishing to deny the importance of this influence, particularly in the case of the French school of mathematics, I nevertheless feel that there has been a tendency to exaggerate it. This has doubtless arisen from the fact that the excellent historical discussion given in Borel's *Leçons sur les Séries Divergentes* has generally been accepted as having wider scope than it actually possesses. The account there given of the effect of the Abel-Cauchy dictum is primarily a description of its influence on the French school, although that fact is not explicitly stated. It is to be expected that any interpretation of Borel's remarks as describing the state of affairs in the mathematical world at large would lead to a somewhat warped view of the situation.

Mathematical science, like all other living things, has its own natural laws of growth. I do not believe that the dictum of any two mathematicians, even of the stature of Abel and Cauchy, is sufficient to ban from consideration any particular branch of investigation that possesses intrinsic importance. It is quite true that the attention paid to divergent series steadily diminished during the first eighty years of the nineteenth century, and that it had almost reached the vanishing point during the last decade of that period. This was due, however, to a variety of factors, of which the Abel-Cauchy dictum was only one element. The gradual development of our present day notions of rigor was perhaps the dominant influence. The general standards of rigor set forth by Abel and Cauchy played a very important role in this development, and thus their indirect influence on the study of divergent series was in itself of considerable importance.

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The extent to which the study of divergent series was continued, both in England and Germany, during the first sixty years of the nineteenth century has been clearly pointed out by Burkhardt in his paper of 1911 in the Mathematische Annalen, *Über den Gebrauch divergenter Reihen in der Zeit von 1750–1860*. Since there was a steady development of notions of rigor during that same period, one may naturally inquire why the rigorous treatment of divergent series did not get an earlier start. I think that the reason is not far to seek. From the point of view of relative frequency and relative simplicity, convergent series bear somewhat the same relationship to series in general that analytic functions do to general functions. Just as it was not feasible for mathematicians to undertake the study of very general types of functions until they had a considerable understanding of the highly special but particularly important class of analytic functions, so it was exceedingly difficult to build a rigorous theory of divergent series before the theory of convergent series had reached some degree of completeness. The attainment of this simpler objective was a task of no mean difficulty, and it absorbed the main energy of students of series during the greater part of the nineteenth century. It is an interesting coincidence that Pringsheim's comprehensive study of convergent series with positive terms was published in the same year (1890) as Cesàro's fundamental paper on the multiplication of series. The latter paper, as we all know, was one of the prime sources of the work of the present century in the field of divergent series.

2. *Origins of the Modern Theory*. Cesàro's paper, published in the Bulletin des Sciences Mathématiques, is just six pages in length and involves only the simplest type of analysis. Nevertheless, if we should take as our criterion of importance for mathematical papers the amount of subsequent literature to which they have led, an examination of the mathematical literature of the first thirty years of the present century would inevitably compel us to rank this paper among the most important contributions of the final quarter of the nineteenth century. At present writing, any such estimate of its importance would appear to be subject to revision upward.

It is also worth while to note, from the standpoint of historical background, that Cesàro clearly understood the scope of the ideas contained in his paper. Toward the close of it we find the following comments:

"Il est téméraire d'affirmer que les séries non convergentes n'auront jamais d'utilité. Tant que cette assertion restera gratuite, nous serons en droit de rechercher sous quelles conditions on peut soumettre les séries indéterminées aux opérations de l'analyse. Après tout, n'est-ce pas en vertu d'une convention que les séries convergentes, prises sous leur forme indéfinie, interviennent dans les calculs?"

We may state, in passing, that in the second edition of Borel's *Leçons sur les Séries Divergentes*, the remark contained in the last sentence of the above quotation is considered to be of recent origin and is ascribed to Knopp. Thus we see again how even the best expository treatments of a subject may serve to disseminate erroneous notions as to its historical development.

The earlier work of Frobenius and Hölder in connection with the generalization of Abel's theorem on the limiting value of a power series to certain types of divergent series has likewise considerable importance in connection with the modern development of the theory of divergent series. The relationship of this work to Cesàro's and its influence on other important studies in the general field of divergent series has already been adequately discussed in various expository treatments of the subject. The most important feature of this work, from the point of view of the present discussion, is that Hölder's general definition and Cesàro's general definition, which are identical for the simple case discussed by Frobenius, were regarded from the first as being of the same general scope, and their complete equivalence was suspected for some time before it was definitely proved.

3. *Divergent Series and the Problem of Analytic Extension.* It was Borel's attack on the problem of analytic extension from the standpoint of the theory of divergent series which first showed clearly that essentially different types of series would in general require essentially different types of summability. The rather obvious necessary condition for either Cesàro or Hölder summability of order r, namely $\lim_{n\to\infty} [u_n/n^r]=0$, at once shows that these methods will be completely ineffective for summing power series outside of their circle of convergence. Nevertheless, Borel found it useful to take Cesàro's notion of mean

value as a point of departure and to generalize it in such a manner as to obtain an effective method for his purpose. This led to the definition known as the Borel mean. A simple transformation yielded the Borel integral definition, which is of somewhat wider scope. The effectiveness of these methods in connection with the study of power series beyond the domain of convergence can be well illustrated by means of the simple series

$$1 + z + z^2 + z^3 + \cdots .$$

We use the integral definition which is set up as follows. Given any series,

$$u_0(z) + u_1(z) + u_2(z) + \cdots ,$$

we define

$$u(z, t) = u_0 + u_1 t + u_2 \frac{t^2}{2!} + \cdots ,$$

and we form

$$\int_0^\lambda e^{-t} u(z, t) dt = \psi(z, \lambda).$$

If $\lim_{\lambda\to\infty} \psi(z, \lambda)$ exists and is equal to $s(z)$, we say that the series is summable (B) to the value $s(z)$. Applying this to the series $\sum z^n$, we have

$$\psi(z, \lambda) = \int_0^\lambda e^{-t} e^{zt} dt = \int_0^\lambda e^{(z-1)t} dt = \left[\frac{e^{(z-1)\lambda}}{z-1} - \frac{1}{z-1}\right],$$

and it is readily seen that $\psi(z, \lambda)$ approaches $1/(1-z)$ as a limit as λ becomes infinite, provided $R(z) < 1$. Therefore the series $\sum z^n$, which diverges everywhere outside the unit circle, will be summable (B) to the value $1/(1-z)$ for all values of z in the half-plane bounded on the right by the perpendicular to the axis of reals at the point $z = 1$.

Thus we see that the introduction of summability (B) adds very considerably to the region in which the series furnishes the value of the generating function. However, it is apparent that there are still some worlds to conquer, and a more powerful method is obviously desirable. This was provided by Leroy by means of the following definition. We set

$$(1) \qquad F(z, t) = \sum_{n=0}^{n=\infty} \frac{\Gamma(nt + 1)}{\Gamma(n + 1)} u_n(z) \qquad (0 < t < 1),$$

and we define the limit as $t \to 1$ of $F(z, t)$ as the sum of the series $\sum u_n(z)$. Let us apply this method to the series $\sum z^n$. Making use of (1) and the following formula for the gamma function

$$\Gamma(nt + 1) = \int_0^\infty e^{-x} x^{nt} dx,$$

we obtain

$$(2) \qquad F(z, t) = \int_0^\infty e^{-x} \sum \frac{(zx^t)^n}{n!} = \int_0^\infty e^{(-x+zx^t)} dx.$$

The path of integration here is along the axis of reals in the x-plane, but this path may be replaced by any other straight line L from the origin to ∞ making an angle less than $\frac{1}{2}\pi$ (in absolute value) with the first path, since the integral along a circular arc joining the two paths approaches zero as the radius becomes infinite.

We can now show by properly choosing the line L, subject to the condition just stated, that the integral in the right-hand member of (2), which converges for $0 < t < 1$ and any fixed value of z, will approach the value $1/(1-z)$ as t approaches 1, for any value of z that does not lie on the axis of reals in the interval $(1 \leqq z < \infty)$, that is for any value of z for which the series $\sum z^n$ does not diverge to $+\infty$. Consider first the behavior of the integral for $t = 1$. If we set $z = \alpha + \beta i$, $x = \rho e^{\phi i}$, it is readily seen that the integral converges to the value $1/(1-z)$, provided z is so chosen that $\alpha \cos \phi - \beta \sin \phi - \cos \phi$ is negative. This condition will be satisfied if the point z lies in a region (T), which is that one of the two half-planes bounded by the line

$$(3) \qquad \alpha \cos \phi - \beta \sin \phi - \cos \phi = 0$$

which contains the origin. We note that the line (3) in the z-plane is the line through the point $z = 1$, parallel to the line symmetric to L with respect to the bisector $\beta = \alpha$.

For any point z in (T) the integral converges uniformly for positive values of $t \leqq 1$; hence the function which it defines, $F(z, t)$, approaches $F(z, 1) = \int_0^\infty e^{x(z-1)} dx = 1/(1-z)$ as t approaches 1. By allowing the line L to take on all admissible positions, the corresponding regions (T) will include any point in the z-plane with the exclusion of points lying in the cut from $z=1$ to $z=\infty$ along the axis of reals.

It would seem at first sight that we should rest content with the result just obtained. We have succeeded in summing the series $\sum z^n$ for all values of z except those for which all the terms are positive and the series diverges to $+\infty$. However, even for such values the series is still the formal expansion of the function $1/(1-z)$ by continued division, by the binomial expansion, or by Taylor's theorem. Why then should it not yield us the sum $1/(1-z)$, if we apply a *suitable* method of summation? This suitable method of summation was furnished by Mittag-Leffler in an article published in volume 42 (1920) of the Acta Mathematica.

We define $E(z)$ by the formula

$$E(z) = \frac{1}{2\pi i} \int^{S} e^{e^\zeta} e^{-e^{e^\zeta}} \frac{d\zeta}{\zeta - z},$$

where the contour S in the ζ-plane is composed of two lines parallel to the axis of reals and extending to the point at infinity on opposite sides of this axis at a distance between $\pi/2$ and $3\pi/2$, together with a line joining these parallels and cutting the axis of reals at an arbitrary point whose abscissa exceeds the real component of z. The function $E(z)$ can be shown to be an integral transcendental function. We now set

$$G(z) = \frac{E(z+b)}{E(b)},$$

where b is real, and we define $K_n(\omega)$ by means of the expansion

$$G[\omega(x-1)] = \sum_{n=0}^{\infty} K_n(\omega) x^n.$$

It can then be shown that if we put $s_n(z) = 1 + z + z^2 + \cdots + z^n$,

$$\lim_{\omega\to\infty} \sum_{n=0}^{\infty} s_n(z) K_{n+1}(\omega) = \frac{1}{(1-z)}$$

for every value of $z \neq 1$, and that the left hand side becomes positively infinite, as ω becomes infinite, for the value $z=1$. We can ask no more of a method for summing the formal expansion of $1/(1-z)$.

The various methods of summation which have been illustrated by means of the simple function $1/(1-z)$ and its corresponding Taylor's series apply to much more general cases. Consider any function, analytic in general but having certain singular points, and its corresponding power series development, $u_0+u_1z+u_2z^2+\cdots$. Through each singular point draw a line perpendicular to the line joining the singular point with the origin. These lines form a polygon, known as the Borel polygon, within which the series is summable by Borel's method to its corresponding function. Draw the infinite half-rays which form the prolongation of the lines drawn from the origin to the singular points. The entire plane, with the exception of these rays, forms a region known as the Mittag-Leffler star. For all points within the star the series is summable to the value of the function by Leroy's method. Finally it is summable by Mittag-Leffler's method to the value of the function at all points of the plane except perhaps the singular points themselves.

4. *Summability of Fourier Series and General Orthogonal Developments.* We have seen that Cesàro's methods are ineffective for the study of power series beyond the circle of convergence. They have proved, on the other hand, to be the methods *par excellence* for the study of Fourier series and other developments in orthogonal functions. Their application to problems of this type dates back to Fejér's fundamental paper of 1903, *Untersuchungen über Fouriersche Reihen.* Fejér's theorem concerning the summability $(C1)$ of Fourier series at all points of continuity or discontinuity of the first kind is now regarded as part of the elementary theory of Fourier series. The fact that the Cesàro means form the ideal method for summing Fourier series is better illustrated, however, by the generalizations of Fejér's theorem due to Lebesgue and Hardy. Let us consider the scope of these results. Any function having a Lebesgue integral over a certain interval serves to define for that interval a unique

Fourier development. The same Fourier development, however, will correspond to any other function differing from the first one at any set of points of measure zero. It is clear then that we cannot in general expect the Fourier development of a function having a Lebesgue integral to furnish the value of the function at all points of the interval, but only for a subset differing from the interval by a set of measure zero. Lebesgue showed in 1905 that we could obtain this maximum result by using summability ($C1$) and Hardy showed in 1913 that we could also obtain it by using summability ($C\rho$) for any $\rho > 0$.

The general state of development of the study of the summability of Fourier series and other expansions in special orthogonal functions that had been reached at the beginning of 1918 was outlined in the writer's Chicago symposium paper, which appeared in volume 25 (1919) of this Bulletin. We shall therefore confine ourselves here to indicating some of the most striking contributions subsequent to that date.

It is well known that the efforts to obtain necessary and sufficient conditions for the convergence of Fourier series have thus far been unsuccessful. The analogous problem for summability by Cesàro means of any particular order is likewise unsolved. To Hardy and Littlewood is due the simple but ingenious idea of formulating a somewhat different problem in the case of Cesàro summability that admits a complete and elegant solution. They considered the situation in which the Fourier series is summable by some Cesàro mean or other, but not necessarily by any particular mean. For this case they obtained the following beautiful result.*

Given a function $f(t)$ that is periodic and has a Lebesgue integral. We set

$$\phi(t) = f(x+t) + f(x-t) - 2A,$$

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(\alpha)d\alpha, \quad \phi_2(t) = \frac{1}{t}\int_0^t \phi_1(\alpha)d\alpha, \cdots .$$

The necessary and sufficient condition that the Fourier development of $f(t)$ should be summable (C) to the value A at the point

* See Mathematische Zeitschrift, vol. 19 (1923), pp. 67–96.

$t=x$ is that there should exist an integer k for which $\phi_k(t)\rightarrow 0$ when $t\rightarrow 0$.

The function $\phi_k(t)$ is expressible in the form

$$\frac{k}{t^k}\int_0^t (t-u)\phi(u)du,$$

and in this form the definition is readily extensible to non-integral values of k. The condition $\phi_k(t)\rightarrow 0$ with t may be conveniently expressed by saying that $\phi(t)\rightarrow O(Ck)$, on account of the analogy between this generalized limit and the Cesàro mean for a series. In addition to their necessary and sufficient condition, Hardy and Littlewood proved the more precise result that if $\phi(t)\rightarrow O(Cr)$ when $t\rightarrow 0$, r being an integer, the Fourier series is summable $(C, r+1)$ to the sum A for $t=x$, and also that if the Fourier series is summable (C, r) to the sum A for $t=x$, then $\phi(t)\rightarrow O$ $(C, r+2)$. They showed somewhat later* that if $\phi(t)\rightarrow O$ $(C\alpha)$ for $0<\alpha<1$, then the series is summable $(C, \alpha+\delta)$ to the sum A at $t=x$ for any positive δ, and also that if the series is summable $(C, -\gamma)$, for $0<\gamma<1$, to the sum A at $t=x$, then $\phi(t)\rightarrow O$ $(C, 1)$. Quite recently Bosanquet has obtained notable generalizations† of these theorems. He extends the first result quoted above to the case of any non-negative α, and he proves the following generalization of the second result. If the Fourier series is summable $(C\alpha)$ to the sum A at $t=x$, for any $\alpha\geqq -1$, then $\phi(t)\rightarrow O$ $(C, \alpha+1+\delta)$ for any positive δ.

In the case of developments in general orthogonal functions Cesàro's methods have likewise proved to be of the greatest practical importance. A series of studies by various authors of the scope of these methods has yielded results of gradually increasing generality. The theory has reached finality in the following very general theorem of Menchoff.‡ For any series of orthogonal functions, $\sum a_n\phi_n(x)$, the condition

$$\sum a_n^2 (\log\log n)^2<\infty$$

is sufficient for summability $(C, k<0)$ almost everywhere in the

* See Proceedings of the Cambridge Philosophical Society, vol. 23 (1927), pp. 681–684.

† See Proceedings of the London Mathematical Society, (2), vol. 31 (1930), pp. 144–164.

‡ See Fundamenta Mathematicae, vol. 8 (1926), pp. 56–108.

interval of definition of the $\phi_n(x)$. For any positive function $\omega(n)$ satisfying the condition $\lim_{n\to\infty} [\omega(n)/(\log\log n)^2]=0$, there exists a series of orthogonal functions for which $\sum a_n\omega_n<\infty$, whereas the series is not summable by any Cesàro mean.

5. *Summability of Dirichlet's Series.* The methods of Cesàro have also been applied to advantage to certain types of Dirichlet's series. For the general Dirichlet's series, $\sum a_n e^{-\lambda_n s}$, we need the more general methods of summation introduced by Marcel Riesz and known as summation by typical means. Representing the general term of the series by u_n, we set

$$C_\lambda{}^\kappa(\omega) = \sum_{\lambda_n<\omega}(\omega-\lambda_n)^\kappa u_n, \quad C_l{}^\kappa(w) = \sum_{l_n<w}(w-l_n)^\kappa u_n,$$
$$(l_n = \log\lambda_n).$$

If the limits $\lim_{\omega\to\infty}[C_\lambda{}^\kappa(\omega)/\omega^\kappa]$, or $\lim_{w\to\infty}[C_l{}^\kappa(w)/w^\kappa]$, exist, we say that the series is summable (R, λ, κ) or (R, l, κ), respectively, to the value of the limit in question. The special cases of the Riesz definition in which $\lambda_n=n$ or $l_n=n$ are entirely equivalent to the Cesàro mean of the same order κ. However, even in this special case, it appears that the Riesz means are better adapted to the study of Dirichlet's series. An analogous situation arises in the case of the developments in Bessel's functions, where equivalent Rieszian means can sometimes be used with more facility than the Cesàro mean. Thus we see that even in the case of two equivalent methods of summation, one of the two may be better adapted to the study of a particular type of series.

The domain of effectiveness of the Riesz typical means in summing Dirichlet's series is in general a half-plane bounded on the left by a line parallel to the axis of imaginaries, although it may be the whole plane or a null region. In the case where we have a half-plane of summability, it does not necessarily follow that the finite portion of the boundary line contains a singular point of the function corresponding to the series. In fact this function may be analytic all over the finite plane, and still the series may be non-summable by any typical mean in a certain half-plane. Our success in extending the domain of summability of power series by using more powerful methods suggests that something analogous may be possible here. That this is actually the case was shown by Riesz in an article that appeared in volume 35 (1912) of the Acta Mathematica.

Riesz took as his point of departure a method of summation used by Mittag-Leffler in connection with power series, which has for such series the same general range of effectiveness as Leroy's method. Adapting this method to the special nature of Dirichlet's series in somewhat the same manner as in the case of his typical means, Riesz sets

$$\phi(\alpha, s) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha\lambda_n + 1)} a_n e^{-\lambda_n s}$$

and defines $\lim_{\alpha\to 0} \phi(\alpha, s)$ as the generalized sum of the series. He shows that for any Dirichlet's series that has a domain of convergence, this method will sum the series to the value of the function corresponding to the series everywhere in the finite plane, except along half-lines drawn through the singular points in the direction of the negative half of the axis of reals. This region is the analogue of the Mittag-Leffler star for power series.

6. *Conclusion.* In conclusion I wish to point out that I have not attempted to give a comprehensive account of the application of various methods of summation to particular types of series. It would be impossible to give such an account within the reasonable bounds of the ordinary expository article. I have aimed rather to select certain high lights of the theory which seem to me to bear out the main thesis of this paper, namely that the ideal type of summation is necessarily different for different types of series. The entire history of the theory furnishes ample warning against a predisposition in favor of any particular method. In approaching the study of any essentially new type of series, our choice of known methods or our construction of a new method must be governed entirely by the inherent nature of the series in question.

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