TWO TYPES OF CONNECTED SETS*

BY P. M. SWINGLE

1. *Definitions.* A set W will be said to be *widely connected* if it is connected† and every connected subset is everywhere dense in it.

A connected set W will be said to be *n-point connected*, where n is any given cardinal number, if there does not exist a subset of power n which disconnects W.

A connected point set (or a continuum) W of type T will be said to be a *perfect connected set* (or a *perfect continuum*) *of type* T if every connected subset (or every subcontinuum) of W is of type T. For example a *perfect one-point connected set* is a connected set, every connected subset of which is one-point connected. A *perfect indecomposable continuum* is a continuum every subcontinuum of which is indecomposable.‡

2. *An Example of a Widely Connected Set.* It will now be shown that under certain logical assumptions, including Zermelo's postulate,§ a widely connected set can exist.

THEOREM 1. *Any bounded indecomposable continuum M, lying in a euclidean space, contains a widely connected subset which is everywhere dense in M.*

Let (K) be the set whose elements are the composants¶ of M, these elements being contained but once in (K). It is known

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† By the notation $W=W_1+W_2$ *separate* is meant that W is the sum of the two non-vacuous, mutually exclusive subsets W_1 and W_2 neither of which contains a limit point of the other. A set W is *connected* if there do not exist subsets W_1 and W_2 such that $W=W_1+W_2$ separate. In this paper a single point will not be considered a connected set. By the notation W' will be meant the set W plus the limit points of W.

‡ See R. L. Wilder, *Characterizations of continuous curves that are perfectly continuous*, Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 614–621.

§ See Alonzo Church, *Alternatives to Zermelo's assumption*, Transactions of this Society, vol. 29 (1927), pp. 178–208.

¶ For definition and properties see Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1, pp. 217–222.

that the power of (K) is that of the linear continuum.* Let c represent this power. Let (C) be the set containing as elements all continua in our space which separate two points of M. It is known that (C) may be taken so that its power is c.†

Let $R(K)$ be a one-to-one correspondence between the elements of (K) and the real numbers (r), $0 \leqq r \leqq 1$; and let $R(C)$ be a one-to-one correspondence between the elements of (C) and these real numbers (r). Let K_r be that element of (K) which corresponds to r of (r) and let C_r be that element of (C) which corresponds to this same r.

As K_r is dense in M,‡ it contains at least one point of C_r. Let w_r be one of these points. Let $W=(w_r)$, then, be the set which contains as elements one and only one point of every K of (K).

It is known that W is connected§ and dense in M. Let Z be any connected subset of W. Assume that Z is not dense in W. Hence Z' is a proper subcontinuum of M and so is contained in a set K of (K). Thus Z is contained in K also. But this is impossible as $W \times K$ contains only one point. Hence Z must be dense in W and so W is widely connected.

It is known that a plane indecomposable continuum M, containing a widely connected subset W, can be projected upon a sphere and then upon a plane S in such a manner that the projection of W on S is an unbounded widely connected set. The following theorem is of interest although its proof is evident.

THEOREM 2. *If W is an unbounded widely connected set, then W does not contain a bounded connected subset.*¶

* S. Mazurkiewicz, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 10, pp. 305–310.

† B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2, p. 253. Indebtedness to this paper for the method of procedure used here is acknowledged as well as to R. L. Wilder, this Bulletin, vol. 33 (1927), pp. 423–427.

‡ Z. Janiszewski and C. Kuratowski, loc. cit., p. 221, Theorem 8.

§ B. Knaster and C. Kuratowski, loc. cit., pp. 233–234, Theorem 37.

¶ See S. Mazurkiewicz, *Sur l'existence d'un ensemble plan connexe ne contenant aucun sous-ensemble connexe, borné*, Fundamenta Mathematicae, vol. 2, pp. 97–103; B. Knaster and C. Kuratowski, loc. cit., p. 244; and G. Poprougénko, *Sur un ensemble connexe plan ne contenant aucune partie connexe bornée*, Fundamenta Mathematicae, vol. 15, pp. 329–336. It is evident that the last two of these examples are not widely connected, as each contains a disconnecting point. Furthermore the first contains such a point as shown by equation 25, p. 100.

3. *Theorems*. The following theorems will hold for any space in which the sets may exist, unless otherwise stated. The truth of the first theorem is evident.

THEOREM 3. *Every widely connected set is a perfect widely connected set.*

LEMMA 1. *If $C-Q$ is disconnected, where Q is finite and C is connected, then C contains a connected subset K and Q contains a point q such that $K-q$ is disconnected and contains $C-Q$.*

THEOREM 4. *Let n be a given positive integer, and W a widely connected set. Then W is a perfect n-point connected set.*

Lemma 1 is seen readily to be true. And Theorem 4 is true. For assume that C is a connected subset of W which is not an n-point connected subset and so contains a finite subset Q of n points which disconnects it. Hence, by Lemma 1, it contains a connected subset K and a point q such that $K-q$ is disconnected. Let $K-q=K_1+K_2$ separate. Then K_1+q is a connected subset of K which is not everywhere dense in it, contrary to Theorem 3. Thus W must be a perfect n-point connected set

THEOREM 5. *If W is a widely connected set, lying in a euclidean space, then W is punctiform.*

Assume that W is not punctiform and so contains a subcontinuum C. But as C lies in a euclidean space it contains a proper subcontinuum contrary to Theorem 3. Hence W is punctiform.

That an n-point connected set is not necessarily punctiform is seen from the following theorem.

THEOREM 6. *Let n be a given positive integer and let M be a perfect indecomposable continuum.* Then M is an n-point connected set.*

For under the assumption that M is not an n-point connected set there exists, by Lemma 1, a connected subset K and a point q such that $K-q=K_1+K_2$ separate. Hence the indecomposable

* For an example of a perfect indecomposable continuum see B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3, pp. 247–286.

continuum K' is the sum of the two proper subcontinua $(K_1+q)'$ and $(K_2+q)'$, which is impossible.

A similar proof establishes the following theorem.*

THEOREM 7. *Let n be a given positive integer. Then a connected subset everywhere dense in an indecomposable continuum is an n-point connected set.*

THEOREM 8. *Let n be a given positive integer and M be a perfect n-point connected set. Then M contains a proper connected subset which disconnects it.*†

For let q_1 and q_2 be any two points of M. Then $(M-q_1)-q_2$ is a connected set such that $M-((M-q_1)-q_2)=q_1+q_2$ separate.

THEOREM 9. *If W is a widely connected set which does not contain a biconnected subset,*‡ *then W is the sum of a countable infinity of mutually exclusive connected subsets.*

As W is not biconnected it is the sum of two mutually exclusive connected subsets, C_0 and U_1, say; likewise U_1 is the sum of two such subsets C_2 and U_2; and $U_i(i=2, 3, \cdots)$ is the sum of two such subsets C_{i+1} and U_{i+1}. Let $C_1=W-(C_2+C_3+\cdots)$. Then as C_1 contains C_0 it is connected and so W is the sum of the mutually exclusive connected subsets $C_1, C_2, \cdots$.

THEOREM 10. *If W is a biconnected subset of an indecomposable continuum M, then W is widely connected if W is dense in M.*

Assume that W is not widely connected, in which case it contains a connected subset C such that C' does not contain W. Suppose that $(W+C)-C'=W-W\times C'=W_1+W_2$ separate. But then the indecomposable continuum M would be the sum of the two proper subcontinua $(W_1+C')'$ and $(W_2+C')'$, as $W'=M$, which is impossible. Thus $W-W\times C'$ is connected. Suppose now that $W-C=Z_1+Z_2$ separate, where Z_1 say con-

* See B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5, p. 37, Corollary, where a related theorem is stated.

† For a study of sets which do not have this property see R. L. Wilder, *On a certain type of connected set which cuts the plane*, Proceedings of the International Congress held at Toronto, vol. 1, 1928, pp. 423–437.

‡ Whether a widely connected set can contain a biconnected subset is a problem of interest in connection with the unsolved problem proposed by C. Kuratowski, Fundamenta Mathematicae, vol. 3, p. 322.

tains the connected set $W-W\times C'$. But since we also know that $M=W'=(W-W\times C')'+C'$, $(W-W\times C')'=M$ and so Z_1 contains Z_2, which is impossible. Therefore $W-C$ is connected and so W is the sum of two mutually exclusive connected subsets, which is a contradiction. Hence W must be widely connected.

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QUADRILATERALS INSCRIBED AND CIRCUMSCRIBED TO A PLANE CUBIC*

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In a paper by M. W. Haskell† the geometrical configurations of triangles inscribing and circumscribing a plane cubic curve have been studied by analytic methods. The purpose of this paper is to examine the properties of quadrilaterals inscribing and circumscribing a plane cubic curve by means of elliptic functions.

The coordinates of a point on the curve will be expressed in terms of Weierstrass' elliptic functions $\wp(u)$ and $\wp'(u)$. It is known that $3n$ points of the cubic are the points of intersection of the cubic with a curve of order n if‡

$$u_1 + u_2 + \cdots + u_{3n} \equiv 0 \mod (\omega_1, \omega_2). \tag{1}$$

The values of the parameters of the vertices of the quadrilaterals are obtained from a consideration of the congruences

$$2u_1 + u_2 \equiv 0,\ \ 2u_2 + u_3 \equiv 0,\ \ 2u_3 + u_4 \equiv 0,\ \ 2u_4 + u_1 \equiv 0,$$

whence

$$15u_1 \equiv 0,$$

or

$$u_1 = \frac{k_1\omega_1 + k_2\omega_2}{15},$$

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† Haskell, this Bulletin, vol. 25 (1918), p. 194.

‡ Appell and Lacour, *Théorie des Fonctions Elliptiques et Applications*, Chap. 3.