EXTENSION OF A THEOREM OF MAZURKIEWICZ*

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S. Mazurkiewicz,† in answer to a question proposed by B. Knaster,‡ has shown that if \( A \) is a closed point set in \( E_n \) (euclidean space of \( n \) dimensions) which is homeomorphic with a subset of \( E_{n-1} \), then all points of \( A \) are accessible from the complementary set, \( E_n - A \). The question naturally arises, then, as to whether the points of \( A \) are regularly § accessible from \( E_n - A \). It will be shown in the present paper that this is indeed the case.

We shall precede our proof by two theorems which, we believe, are themselves of fundamental importance. Following Mazurkiewicz' notation, we shall denote by \( S_n(p, \rho) \) a spherical neighborhood of a point \( p \) of \( E_n \) with radius \( \rho \); by \( \phi(A) \), the subset of \( E_{n-1} \) that is homeomorphic with \( A \); and if \( X \) is any subset of \( A \), by \( \phi(X) \) we denote that subset of \( \phi(A) \) that corresponds to \( X \) under the homeomorphism between \( A \) and \( \phi(A) \). Also, following the usual custom, if \( M \) is a point set, by \( \overline{M} \) we shall denote the set \( M \) together with all of its limit points.

Evidently the proof given by Mazurkiewicz for his Lemme establishes the following more general lemma.

**Lemma 1.** Let \( P \) be a point of \( A \), \( D \) a domain¶ containing \( P \), and \( G \) a component of \( D - A \cdot D \) such that \( \overline{G} \ni P \). Then, if \( D_1 \) is a bounded domain such that \( D_1 \subset D \) and \( D_1 \ni P \), there is a component \( G_1 \) of \( G \cdot D_1 \) such that \( P \in \overline{G_1} \).

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‡ See Fundamenta Mathematicae, vol. 8 (1926), Problem 43, p. 376.
§ A point \( P \) of a point set \( M \) is said to be regularly accessible from a point set \( R \) of which \( P \) is a limit point provided that for every \( \epsilon > 0 \) there exists a positive number \( \delta \) such that if \( Q \) is a point of \( R \) whose distance from \( P \) is less than \( \delta \), then there is an arc from \( P \) to \( Q \) whose diameter is less than \( \epsilon \) and which lies, except for \( P \), wholly in \( R \). See G. T. Whyburn, this Bulletin, vol. 34 (1928), p. 509.
¶ By domain we mean a connected open subset of the space under consideration. The domain \( D \) may, of course, be \( E_n \), in which case the component \( G \) of this lemma will necessarily exist, due to the invariance of dimensionality under analysis situs transformations.
**Theorem 1.** If $D$ is a bounded domain of $E_n$, and a component $C$ of $A \cdot D$ separates* $D$, then the set $\phi(C)$ is a domain of $E_{n-1}$ whose boundary is $\phi(C - C)$.

**Proof.** Since, due to similarities between the proof and that given by Mazurkiewicz for his Lemme, we can conserve space by referring to the latter, we shall endeavor to retain most of his notation in this connection.

We can assume that $A = C$; then the set $\phi(A)$ does not fill up $E_{n-1}$. Hence $E_n - A$ is connected,† and if $a$ is a point of $C$ there exists, by the above Lemma, a component $G$ of the set

$$(E_n - C) \cdot D$$

such that $G \ni a$. Let $B = A \cdot F = C - C$,‡ where $F$ is the boundary of $D$, and let $H$ be the component of $E_{n-1} - \phi(B)$ that contains $\phi(a)$. We shall show that $H = \phi(C)$.

Since $C$ separates $D$, there are two points, $c$ and $c_1$, in (1), which do not lie in the same component of (1); we may suppose that $G \ni c$. Then $G - G$ is a cut of $E_n$ between $c$ and $c_1$, and accordingly contains an irreducible cut, $L$, of $E_n$, between $c$ and $c_1$. Let

$$L = L_1 + L_2,$$

where

$$(2') L_1 = L \cdot A, \quad L_2 = L \cdot F.$$  

It is clear that $L_1 \cdot D \neq 0$, and hence $L \cdot C \neq 0$, all points of $A$ in $D$ belonging to $C$. We now note that

$$\phi(C) \subset H,$$

since $\phi(C) \cdot \phi(B) = 0$ and $\phi(C) \ni \phi(a)$. Also, that

$$\phi(L_1) \cdot H \subset \phi(C),$$

since

$$\phi(L_1) \cdot H \subset \phi(A) \cdot H \subset [\phi(B) + \phi(C)] \cdot H = \phi(C) \cdot H = \phi(C).$$

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* That is, there exist, in $D - C$, two points $P$ and $Q$ which are not joined by any subcontinuum of $D - C$.


‡ That $B \neq 0$ is an immediate consequence of the Alexander Addition Theorem. (See proof of Theorem 2 below.)
Theorem of Mazurkiewicz

Suppose that \( H \) contains a point which is not in \( \phi(L_1) \). Then we can express \( H \) as the sum of two mutually exclusive sets, \( H_1 \) and \( H_2 \), where

\[
H_1 = H \cdot \phi(L_1), \quad H_2 = H - H_1.
\]

Since \( \phi(L_1) \) is closed, it follows from (5) that \( H_1 \) is closed in \( H \). Then, since \( H \) is connected, \( H_1 \) contains a limit point, \( \phi(P) \), of \( H \).

Let \( \epsilon \) be a positive number less than \( \rho[\phi(P), \phi(B)] \), and such that \( F_{n-1}[\phi(P), \epsilon] \) contains a point, \( Q \), of \( H \), that is not in \( \phi(L_1) \). Such a point exists, of course, since \( P \) is a limit point of the set of such points. Let

\[
\phi(L_1) \cdot S_{n-1}[\phi(P), \epsilon] = \phi(L_3),
\]

\[
\phi(L_1) \cdot \{E_{n-1} - S_{n-1}[\phi(P), \epsilon]\} = \phi(L_4).
\]

That \( \phi(L_3) \neq 0 \) is obvious, since \( \phi(P) \subset \phi(L_3) \). That \( \phi(L_4) \neq 0 \) follows from the following considerations. If we denote the set of points of \( L \) that are not in \( L_2 \) by \( L' \), then \( L_2 \) contains a limit point, \( x \), of \( L' \). Since \( L' \subset C \), the point \( x \) is contained in \( \overline{C - C} = B \). By a theorem of Miss Mullikin, the continuum \( L \) contains a connected set, \( L'' \), which contains \( P \) and has at least one limit point in \( B \), but contains no point of \( B \). It is easy to see that \( L'' \subset L_1 \), and consequently that \( \phi(L'') \) must have points in \( F_{n-1}[\phi(P), \epsilon] \) that are also points of \( \phi(L_1) \). Thus \( \phi(L_4) \neq 0 \).

Since \( \phi(L_3) \cdot \phi(L_4) \subset F_{n-1}[\phi(P), \epsilon] \), and since \( Q \) is a point of \( F_{n-1}[\phi(P), \epsilon] \) that is not in \( \phi(L_1) \), it is clear that \( \phi(L_3) \cdot \phi(L_4) \) does not fill up the surface \( F_{n-1}[\phi(P), \epsilon] \), and consequently that the \((n-2)\)th Betti number \((\text{mod } 2)\) of \( \phi(L_3) \cdot \phi(L_4) \) is zero. In symbols,

\[
p^{n-2}[\phi(L_3) \cdot \phi(L_4)] = 0.
\]

By (2), (6) and (7),

\[
L = L_1 + L_2 = L_3 + (L_2 + L_4).
\]

* Anna Mullikin, *Certain theorems relating to plane connected point sets*, Transactions of this Society, vol. 24 (1922), pp. 144-162.

Neither of the sets $L_3$, $L_2 + L_4$, is identical with $L$; for $L_2 + L_4$ does not contain $P$, and $L_3$ contains no point of $L_2$. Consequently, since $L$ is an irreducible cut between $c$ and $c_1$,

$$c + c_1 \sim 0 \quad \text{(mod } 2, E_n - L_3),$$

$$c + c_1 \sim 0 \quad \text{[mod } 2, E_n - (L_2 + L_4)].$$

Now

$$(11) \quad L_3 \cdot (L_2 + L_4) = L_2 \cdot L_3 + L_3 \cdot L_4 = L_3 \cdot L_4.$$ 

Since $L_3 \cdot L_4$ is homeomorphic with $\phi(L_3) \cdot \phi(L_4)$, and the Betti number of a closed set is an analysis situs invariant, it follows from (8) and (11) that

$$(12) \quad p^{n-2}[L_3 \cdot (L_2 + L_4)] = p^{n-2}(L_3 \cdot L_4) = 0.$$ 

Consequently, by virtue of Alexandroff's generalization of the Phragmén-Brouwer theorem,* and relations (9), (10), and (12),

$$c + c_1 \sim 0 \quad \text{(mod } 2, E_n - L).$$

But this is a contradiction of the fact that $L$ is a cut of $E_n$ between $c$ and $c_1$. Thus the supposition that $H$ contains a point not in $\phi(L_1)$ leads to a contradiction, and

$$(14) \quad H \subseteq \phi(L_1).$$

From relations (3) and (14) we have that

$$(15) \quad \phi(C) \subseteq \phi(L_1),$$

and hence, from (3) and (15),

$$(16) \quad \phi(C) \subseteq H \cdot \phi(L_1).$$

From relations (4), (14), and (16) it follows that

$$\phi(C) = H \cdot \phi(L_1) = H.$$ 

As $H$ is an open connected subset of $E_{n-1}$, the theorem is proved, the relations $\overline{H} - H = \phi(\overline{C} - C) = \phi(B)$ being an immediate consequence of the fact that $H = \phi(C)$.

**Theorem 2.** In $E_n$, let $D$ be a bounded domain such that all

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1-cycles of $D$ are homologous to zero in $D$.\* Then if two points $c_1$ and $c_2$ of $D$ are separated in $D$ by $A \cdot D$, there is a component, $C$, of $A \cdot D$ which separates $c_1$ and $c_2$ in $D$.

**Proof.** Let that component of $D - A \cdot D$ which contains $c_1$ be denoted by $G$. Then $G - G$ is a cut of $E_n$ between $c_1$ and $c_2$, and contains an irreducible cut, $L_i$, between $c_1$ and $c_2$. Let $L_1$ and $L_2$ be defined as in (2') above. Let $P$ be a point of $L_1 \cdot D$, and let $C$ be the component of $A \cdot D$ determined by $P$. Let $B = A \cdot F$, where $F$ is the boundary of $D$, and let $H$ be the component of $E_{n-1}$ $- \phi(B)$ determined by $\phi(P)$. As before, if we suppose $H$ contains a point which is not in $\phi(L_1)$, we can separate $H$ according to relations (5), and proceed to a contradiction; consequently $H \subset \phi(L_1)$. Since $\phi(L_1) \subset \phi(A)$, we have $H \subset \phi(A)$ and accordingly $\phi^{-1}(H) \subset A$. Since we know that $H$ contains no point of $\phi(B)$, $\phi^{-1}(H) \subset A - B$. Therefore $\phi^{-1}(H) \subset C$. It is clear that $\phi(C) \subset H$, and therefore $C \subset \phi^{-1}(H)$. Consequently $\phi^{-1}(H) = C$ and $H = \phi(C)$. Since, as noted above, $H \subset \phi(L_1)$, it follows that $C$ is the component of $L_1 \cdot D$ determined by $P$. Thus every component of $L_1 \cdot D$ is homeomorphic with an $(n-1)$-dimensional domain of $E_{n-1}$ whose boundary is in $\phi(B)$.

Let $t$ be any 1-chain in $D$ bounded by $c_1 + c_2$. Not more than a finite number of the components of $L_1 \cdot D$ contain points of $t$. For suppose infinitely many contain points of $t$, and let $C_1, C_2, C_3, \ldots$ denote these components; they form a denumerable collection, since, as just shown, every component of $L_1 \cdot D$ is homeomorphic with a domain of $E_{n-1}$, and no two components have points in common. Let $x_i$ be a point of $C_i \cdot t$, $(i = 1, 2, 3, \ldots)$. Then the set $\sum i=1^{\infty} x_i$ has at least one limit point, $y$, on $t$. As $L_1$ is closed, $y \in L_1$, and there is a component, $U$, of $L_1 \cdot D$ that contains $y$. But $\phi(y)$ is an interior point of the domain $\phi(U)$ and cannot be a limit point of the set $\sum i=1^{\infty} \phi(x_i)$. The contradiction

\* We refer here to modulo 2 homologies. See J. W. Alexander, Combinatorial analysis situs, Transactions of this Society, vol. 28 (1926), pp. 301-329. The necessity for this condition on the 1-cycles is made evident by the case where $D$ is the interior of the anchor-ring in $E_3$; for a plane may be passed through $D$, in this case, in such a way that two points of $D$ are separated by two components of the plane section, but not separated by either one of the components. An important case where the condition is satisfied is of course that in which $D$ is bounded by the topological $(n-1)$-sphere.
is obvious. Let those components of $L_1 \cdot D$ that have points in common with $t$ be denoted by $K_1, K_2, \ldots, K_m$.

One of the components $K_i$ separates $c_1$ from $c_2$ in $D$. To show this, we note first that their sum $\sum_{i=1}^m K_i$ separates $c_1$ from $c_2$ in $D$. For suppose this is not the case. Then, denoting the set $L_1 \cdot D - \sum_{i=1}^m K_i$ by $L_i'$, there exist 1-chains $T_1' (\neq t, \text{say})$ and $T_2'$, such that*

$$\begin{align*}
T_1' & \to c_1 + c_2, \quad [D - L_i'], \quad [E_n - (L_2 + L_i')], \\
T_2' & \to c_1 + c_2, \quad [D - \sum_{i=1}^m K_i], \quad [E_n - \left( L_2 + \sum_{i=1}^m K_i \right)].
\end{align*}$$

(17)

The sets $L_2 + L_i'$ and $L_2 + \sum_{i=1}^m K_i$ are closed, and have in common only $L_2$. The latter set, however, lies entirely in $F$. Accordingly, by the hypothesis, there exists a 2-chain, $T^2$, bounded by $T_1' + T_2'$ in $D$, and we have

$$T_1' + T_2' \sim 0, \quad \left[ E_n - (L_2 + L_i') \cdot \left( L_2 + \sum_{i=1}^m K_i \right) \right].$$

(18)

By Alexander's Addition Theorem,† and by (17) and (18),

$$c_1 + c_2 \sim 0, \quad (E_n - L).$$

(19)

But (19) contradicts the fact that $L$ is a cut of $E_n$ between $c_1$ and $c_2$. Consequently one of the sets $L_i', \sum_{i=1}^m K_i$, separates $c_1$ and $c_2$ in $D$, and as the former set has no points in $t$, it is obvious that $c_1$ and $c_2$ are separated by $\sum_{i=1}^m K_i$ in $D$.

The proof can now be completed by a finite number of steps. If $K_i$ does not separate $c_1$ and $c_2$ in $D$, we can show by use of the Alexander Addition Theorem that $\sum_{i=1}^m K_i$ separates $c_1$ and $c_2$ in $D$. By process of elimination we must finally arrive at a set $K_j, (1 \leq j \leq m)$, which separates $c_1$ and $c_2$ in $D$. As $K_j \subset L_1$ and $L_1 \subset A$, the component $K_j \subset A$ and the theorem is proved.

**Theorem 3.** Let $P$ be a point of $A$. Then for any positive number $\rho$ there exists a positive number $\epsilon$ such that if $Q$ is any point of

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* If $C^i$ denote an $i$-cycle, then the relation $M^{i+1} \to C^i$ is to be interpreted "$M^{i+1}$ is an $(i+1)$-chain bounded by $C^i"$. See J. W. Alexander, loc. cit. All congruences and homologies used in the present instance are to be understood as modulo 2, without explicit statement of that fact in the relations given.

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\( (E_n - A) \cdot S_n(P, \epsilon) \), then that component of \( (E_n - A) \cdot S_n(P, \rho) \) which contains \( Q \) has \( P \) as a boundary point.

**Proof.** Suppose there exists a positive number \( \rho \) for which the theorem is not true. Let \( \epsilon_1 \) be a positive number less than 1 as well as less than \( \rho \). Then the neighborhood \( S_n(P, \epsilon_1) \) contains a point, \( x_1 \), of \( E_n - A \), such that the component, \( G_1 \), of \( (E_n - A) \cdot S_n(P, \rho) \) which contains \( x_1 \) does not have \( P \) as a boundary point. Then \( \overline{G_1} - G_1 \) is a cut of \( E_n \) between \( x_1 \) and \( P \). Accordingly, \( (\overline{G_1} - G_1) \cdot S_n(P, \rho) \) is a cut of \( S_n(P, \rho) \) between \( x_1 \) and \( P \), and by virtue of Theorem 2 (as applied to the closed set \( (\overline{G_1} - G_1) \cdot A \)) there is a component, \( C_1 \), of this set, which separates \( x_1 \) and \( P \) in \( S_n(P, \rho) \). By Theorem 1 the set \( \phi(C_1) \) is a domain of \( E_{n-1} \) whose boundary is \( \phi(\overline{C_1} - C_1) \c F_n(P, \rho) \). It is clear, then, that \( C_1 \) is a component of \( A \cdot S_n(P, \rho) \).

Let \( \epsilon_2 \) be a positive number less than \( \frac{1}{2} \) as well as less than \( \epsilon_1 \) and \( \rho(P, C_1) \). Then \( S_n(P, \epsilon_2) \) contains a point \( x_2 \) of \( E_n - A \) such that the component, \( G_2 \), of \( (E_n - A) \cdot S_n(P, \rho) \) which contains \( x_2 \) does not have \( P \) as a boundary point. As before, there is a component, \( C_2 \), of \( A \cdot S_n(P, \rho) \) which separates \( x_2 \) from \( P \) in \( S_n(P, \rho) \).

Continuing in this way, we obtain a sequence of distinct points \( x_1, x_2, x_3, \ldots \), having \( P \) as a sequential limit point, and a sequence \( C_1, C_2, C_3, \ldots \) of distinct components of \( A \cdot S_n(P, \rho) \) such that for every \( i, x_i \) and \( P \) are separated in \( S_n(P, \rho) \) by \( C_i \).

From the fact that \( x_i \in S_n(P, \epsilon_i) \) and \( C_i \) separates \( x_i \) and \( P \) in \( S_n(P, \rho) \), it follows that there is a point \( y_i \) of \( C_i \) in \( S_n(P, \epsilon_i) \).

That the sequence \( y_1, y_2, y_3, \ldots \) has \( P \) as a sequential limit point is obvious. Let

\[
(20) \quad \overline{C_i} \cdot F_n(P, \rho) = B_i, \quad (i = 1, 2, 3, \ldots).
\]

Then \( B_i \subset A \), and by Theorem 1, \( \phi(C_i) \) is a domain of \( E_{n-1} \) whose boundary is \( \phi(B_i) \).

For every \( i \), \( \phi(B_i) \) separates \( \phi(y_i) \) from \( \phi(P) \) in \( E_{n-1} \), and hence \( \phi(P) \) is a limit point in \( E_{n-1} \) of the set \( \sum_{i=1}^{\epsilon} \phi(B_i) \). But then in \( E_n \), \( P \) must be a limit point of the set \( \sum_{i=1}^{\epsilon} B_i \), which is absurd since by (20) the sets \( B_i \) are all in \( F_n(P, \rho) \).

The following theorem now follows simply from Theorem 3.

**Theorem 4.** Every point of \( A \) is regularly accessible from \( E_n - A \).

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