# ON THE JOIN OF TWO COMPLEXES*

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1. *Introduction.* Notations are as in Lefschetz's colloquium Publication.‡ Where not otherwise specified, cells and spheres are combinatorial.

In Chapter II of Lefschetz I the join of two complexes is defined, and it is proved that the join of two cells, or of a cell and a sphere, is a cell. We shall prove that if the given cells are normal, the join is likewise normal. In the later part of the paper we obtain formulas for the Betti numbers of the join of any two complexes.

2. *The Join of two Cells.*

THEOREM 1. *The join of the closures of two normal cells is the closure of a normal cell.*

PROOF. Since Lefschetz proves§ that the join is a cell, it remains only to prove that it is normal, that is, the join of a point and a sphere.

Let $A$ and $B$ denote the closed cells. Since they are normal, we may consider them to be located in a euclidean $m$-space; composed of simplexes; each having only one interior vertex, called $C_a$ and $C_b$ respectively; and such that $(A, B)$ consists of $A$, $B$ and the points on non-intersecting line segments, called *elements*, joining the points of $A$ to the points of $B$.

In Lefschetz I, pp. 112–113, it is shown that $(A, F(B))$ and $(F(A), B)$, locus of the chain boundary of $(A, B)$, constitute a sphere. We shall call them the *boundary* of $(A, B)$, and the remaining points the *interior* of $(A, B)$. Let $C$ be an interior point of $(C_a, C_b)$. We shall prove that $(A, B)$ is the join of $C$ with the boundary.

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† Part of the work on this paper was done while the author was a National Research Fellow, at Princeton University.

‡ S. Lefschetz, *Topology*, New York, 1930. (Lefschetz I.)

§ Lefschetz I, pp. 111–112.

Let $P$ be any point on $(A, B)$ not $C$ itself. We shall prove: (1) interior points of $(P, C)$ are interior points of $(A, B)$; (2) if $P$ is an interior point of $(A, B)$, then $(C, P)$ can be continued in a straight line beyond $P$ in the interior of $(A, B)$.

CASE I: $P$ is an interior point of $(P_a, P_b)$, where $P_a$ and $P_b$ are points of $A$ and $B$ distinct from $C_a$ and $C_b$, respectively. Now $C_a$, $C_b$, $P_a$, $P_b$ must determine a non-degenerate tetrahedron, since if they lay in any 2-plane we could find two non-identical elements having a common interior point. From the construction of a tetrahedron as the join of either pair of opposite edges, it follows that interior points of $(C, P)$ are interior points of the tetrahedron, consequently are interior points of joins of pairs of points one on the interior of each of the segments $(C_a, P_a)$ and $(C_b, P_b)$. Since such points are interior points of $(A, B)$, we conclude that (1) is proved.

In case $P$ is an interior point of $(A, B)$, then $P_a$ and $P_b$ must be interior points of $A$ and $B$, respectively. Hence we can find points $P_a'$ and $P_b'$ on $A$ and $B$, respectively, such that $P_a$ and $P_b$ are interior points of $(C_a, P_a')$ and $(C_b, P_b')$, respectively. Then $P$ will be an interior point of the tetrahedron determined by $C_a$, $P_a'$, $C_b$, $P_b'$; and since $(C, P)$ consists of points in the interior of that tetrahedron, we conclude that (2) is valid.

CASE II: All other cases. The proofs here are simpler, due to the fact that we have to deal with triangles at worst, instead of tetrahedrons as in the case just considered. Hence we omit the details.

From (1) we conclude that the interior points of the segments joining $C$ to points of the boundary of $(A, B)$ are themselves points of $(A, B)$, and none of them is obtained more than once by the process. From (2) and the fact that $(A, B)$ is in a bounded part of $m$-space, we infer that all the points of $(A, B)$ are obtained in this way, that is, on joins of $C$ with points of the boundary of $(A, B)$. Since this boundary is a sphere, it follows that $(A, B)$ is the closure of a normal cell; that is, Theorem 1 is proved.

## 3. *The Join of a Cell and a Sphere.*

THEOREM 2. *The join of a sphere and the closure of a normal cell is the closure of a normal cell.*

PROOF. Let $S$ and $E$ denote the sphere and the closed cell, respectively. As above, we take them composed of simplexes in a euclidean $m$-space, such that $E$ has only one interior vertex and $(S, E)$ consists of the points on the line segments joining points of $S$ with points of $E$. The locus of the chain boundary of $(S, E)$ is $(S, F(E))$, a sphere,* which we shall call the *boundary* of $(S, E)$. All other points of $(S, E)$ will be called *interior* points.

Let $C$ be the interior vertex of $E$, and $P$ any point of $(S, E)$ other than $C$ itself. From this point the proof continues as in the case of Theorem 1. In Case I, $P$ is an interior point of $(P_s, P_e)$, where $P_s$ is a point on $S$ and $P_e$ is a point on $E$ distinct from $C$. In Case II, $P$ is either on the join of $C$ with a point of $S$, or on $E$. In both cases the proofs are simpler than the corresponding proofs for Theorem 1, involving triangles at worst. We shall give no further details.

4. *Betti Numbers of the Join of two Complexes.*

THEOREM 3. *Given complexes $A$ and $B$, let $A_i$, $B_i$, $P_i$, $J_i$ denote the ith Betti numbers of $A$, $B$, their product and their join, respectively. Then*†

$$J_i = P_{i-1} - B_{i-1} - A_{i-1} + \delta_{i-1}^{0}, \quad i \geqq 1;$$

$$J_0 = 1.$$

*These formulas hold also for Betti numbers* mod $p$, *$p$ any prime greater than unity.*

PROOF. Let $A$ and $B$ be regularly subdivided, and images composed of simplexes taken in a euclidean space so that $(A, B)$ is obtained by joining the points of $A$ to the points of $B$ by straight line segments, called elements. We introduce an additional coordinate, say $z$, keeping the other coordinates of the points of $A$ and $B$ fixed, and place $A$ in the hyper plane $z=1$ and $B$ in the plane $z=-1$. Let $C$ denote the intersection of $(A, B)$ with the plane $z=0$.

Let $A_1$ denote the part of $(A, B)$ for which $z\geqq 0$, and $B_1$ the

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* Lefschetz I, p. 112.

† $\delta_j^i=1$ or 0 according as $i=j$ or $i\neq j$.

part for which $z \leqq 0$. Then we have, from Lemmas 1, 2, 4 and 6 in an earlier paper,* the following equalities:

$$R_i(A_1) = a^i + c_1{}^i + c_b{}^i; \tag{1}$$

$$R_i(B_1) = b^i + c_1{}^i + c_a{}^i; \tag{2}$$

$$R_i(C) = c_1{}^i + c_a{}^i + c_b{}^i + c_d{}^i; \tag{3}$$

$$R_i(A, B) = a^i + b^i + c_1{}^i + c_d{}^{i-1}. \tag{4}$$

Now $a^i$ is the maximum number of $i$-cycles on $A_1$ that are independent† of the $i$-cycles on $C$. Any $i$-cycle on $A_1$ is homologous to one on $A$, since it can be deformed onto $A$ along the elements. We can then deform the resulting $i$-cycle along the elements joining $A$ to a single point of $B$, as far as the locus $z=0$, that is, up to $C$. Consequently every $i$-cycle on $A_1$ is homologous to one on $C$. Therefore $a^i=0$. By symmetry, $b^i=0$.

Any $i$-cycle on $C$ can be deformed first onto $A$ along the elements, then to any point of $B$ along the elements joining $A$ to that point of $B$. Since $c_1^i$ is the maximum number of $i$-cycles on $C$ which are independent on $D$, we conclude that $c_1^i=0$, for $i \geqq 1$. Since $(A, B)$ is connected, $c_1^0=1$.

The locus $C$ is a homeomorph of the product of $A$ and $B$. Hence $R_i(C)=P_i$, under our notation.‡ Since $A_1$ can be deformed onto $A$ along the elements, it follows from Theorem 2§ of our paper cited above that $R_i(A_1)=R_i(A)$. By symmetry, $R_i(B_1)=R_i(B)$.

By substituting the values just obtained in (1), (2) and (3), and then substituting from (1), (2) and (3) in (4), we obtain the first relation of Theorem 3. The second is a consequence of the fact that $(A, B)$ is connected.

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* A. B. Brown, *Relations between the critical points of a real analytic function of n independent variables*, American Journal of Mathematics, vol. 52 (1930), pp. 251–270. The lemmas in question, stated for Betti numbers absolute and mod 2, hold also for Betti numbers mod $p$, where $p$ is any prime greater than unity; as do the proofs. The meanings of some of the symbols are explained below.

† Independence refers to homologies.

‡ See Lefschetz I, Chap. 5. $P_i = \sum_{r+s=i} A_r B_s$.

§ This theorem holds for Betti numbers mod $m$, $m$ any integer greater than unity. The proof is easily modified to cover this case.