ON THE NUMBER OF APPARENT DOUBLE POINTS OF r-SPACE CURVES

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Consider a curve C^N of order N in r-space. The number, h, of $(r-2)$-spaces passing through a given $(r-3)$-space and meeting C^N twice is finite. If C^N is projected on to a 3-space, then h is the number of apparent double points on the projection. To avoid circumlocution, we shall use the phrase *the apparent double points of* C^N instead of *the apparent double points of the* 3-*space projection of* C^N.

When the curve C^N is the intersection of $r-1$ hypersurfaces of order $n_1, n_2, \cdots, n_{r-1}$, the number of its apparent double points is known and is given by the formula*

$$(1) \qquad h = \tfrac{1}{2}n_1n_2\cdots n_{r-1}(n_1n_2\cdots n_{r-1} - \textstyle\sum n_i + r - 2).$$

But suppose C^N is not the intersection of $r-1$ hypersurfaces but the intersection of $q<r-1$ varieties $V_{r_1}^{n_1}, V_{r_2}^{n_2}, \cdots, V_{r_q}^{n_q}$ of orders $n_1, n_2, \cdots, n_q$ and of dimensions (which may be different) $r_1, r_2, \cdots, r_q$ where

$$(2) \qquad r_1 + r_2 + \cdots + r_q = r(q-1) + 1.$$

What is the formula for h for such a curve? It is our purpose in this paper to derive this formula.

As a first step in the derivation, let $q=2$. Then C^N or $C^{n_1n_2}$ is the intersection of two varieties $V_{r_1}^{n_1}$, $V_{r_2}^{n_2}$, where $r_1+r_2=r+1$. Let h_i be the number of apparent double points on the curve C^{n_i} in which an S_{r_i} meets $V_{r_i}^{n_i}$. Decompose one of the given varieties, say $V_{r_1}^{n_1}$, into n_1 r_1-spaces having severally $\frac{1}{2}n_1(n_1-1)-h_1$ (r_1-1)-spaces in common. The curve C^{n_1} in which an S_{r_2} meets the decomposed $V_{r_1}^{n_1}$ is, then, composed of n_1 lines forming a skew n_1-sided polygon with $\frac{1}{2}n_1(n_1-1)-h_1$ vertices. Now the curve $C^{n_1n_2}$ in which $V_{r_2}^{n_2}$ meets the decomposed $V_{r_1}^{n_1}$ is composed of n_1

* Veronese, *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882), pp. 161–234. The formula above is given on p. 205.

curves all of order n_2. If any two of these n_1 curves intersect, they must intersect in n_2 points lying in one of the $\frac{1}{2}n_1(n_1-1)-h_1$ $(r-1)$-spaces mentioned above. Each of these (r_1-1)-spaces contains n_2 such points. Hence, the total number of points in which the n_1 curves actually intersect severally is seen to be $n_2[n_1(n_1-1)/2-h_1]$. The total number of intersections, both actual and apparent, of the n_1 curves two by two is $\frac{1}{2}n_1n_2^2(n_1-1)$. Now each of the n_1 curves has h_2 apparent double points. Therefore, we conclude that the number h of apparent double points on the curve $C^{n_1n_2}$, proper or improper, is equal to the sum of the number of apparent intersections of the component curves of the degenerate $C^{n_1n_2}$ and the total number of the apparent double points on the component curves, that is,

$$(3)\qquad \begin{aligned} h &= \tfrac{1}{2}n_1n_2^2(n_1-1) - n_2[n_1(n_1-1)/2 - h_1] + n_1h_2 \\ &= \tfrac{1}{2}n_1n_2(n_1n_2 - n_1 - n_2 + 1) + n_2h_1 + n_1h_2. \end{aligned}$$

Suppose we have a curve $C^{n_1n_2n_3}$ which is the intersection of three varieties $V_{r_1}^{n_1}$, $V_{r_2}^{n_2}$, $V_{r_3}^{n_3}$ in S_{r_i}, where $r_1+r_2+r_3=2r+1$. Let h_i be the number of apparent double points on the curve C^{n_i} in which an S_{r-r_i+1} meets $V_{r_i}^{n_i}$. To find the number h of apparent double points on $C^{n_1n_2n_3}$, we may reason as above or we may proceed as follows.

The curve $C^{n_1n_2n_3}$ may be considered as the intersection of $V_{r_3}^{n_3}$ and the variety $V_{r_1+r_2-r}^{n_1n_2}$, the latter being the intersection of $V_{r_1}^{n_1}$ and $V_{r_2}^{n_2}$. Let h_{12} be the number of apparent double points on the curve $C^{n_1n_2}$ in which an S_{r_3} meets $V_{r_1+r_2-r}^{n_1n_2}$ and its value is given by (3). Applying formula (3), we find, replacing n_1, n_2, h_1, h_2 by n_1n_2, n_3, h_{12}, h_3 respectively,

$$h = \tfrac{1}{2}n_1n_2n_3(n_1n_2n_3 - n_1n_2 - n_3 + 1) + n_3h_{12} + n_1n_2h_3.$$

Writing for h_{12} its value from (3) in the above, we obtain

$$(4)\quad h = \tfrac{1}{2}n_1n_2n_3(n_1n_2n_3 - n_1 - n_2 - n_3 + 2) + n_2n_3h_1 + n_3n_1h_2 + n_1n_2h_3$$

as the number of apparent double points on $C^{n_1n_2n_3}$.

Now let $q=4$. Then C^N, where $N=n_1n_2n_3n_4$, is the intersection of four varieties $V_{r_1}^{n_1}$, $V_{r_2}^{n_2}$, $V_{r_3}^{n_3}$, $V_{r_4}^{n_4}$, where $r_1+r_2+r_3+r_4=3r+1$. We may regard C^N as the intersection of $V_{r_4}^{n_4}$ and the variety $V_{r_1+r_2+r_3-2r}^{n_1n_2n_3}$, the latter being the intersection of $V_{r_1}^{n_1}$, $V_{r_2}^{n_2}$, $V_{r_3}^{n_3}$, and apply (3) and (4), or we may regard it as the intersection of a $V^{n_1 n_2}_{r_1+r_2-r}$ and a $V^{n_3 n_4}_{r_3+r_4-r}$, the former being the intersection of $V^{n_1}_{r_1}$, $V^{n_2}_{r_2}$ and the latter that of $V^{n_3}_{r_3}$, $V^{n_4}_{r_4}$, and then apply (3) alone. Adopting the latter view, we have, replacing n_1, n_2, h_1, h_2 by $n_1 n_2$, $n_3 n_4$, h_{12}, h_{34} respectively in (3),

$$h = \tfrac{1}{2} n_1 n_2 n_3 n_4 (n_1 n_2 n_3 n_4 - n_1 n_2 - n_3 n_4 + 1) + n_1 n_2 h_{34} + n_3 n_4 h_{12},$$

where h_{12} and h_{34} are the respective numbers of apparent double points on the curves $C^{n_1 n_2}$, $C^{n_3 n_4}$ in which an $S_{r_3+r_4-r}$ and an $S_{r_1+r_2-r}$ meet the varieties $V^{n_1 n_2}_{r_1+r_2-r}$ and $V^{n_3 n_4}_{r_3+r_4-r}$ respectively. Now h_{12} is given by (3) and h_{34} is also given by (3) if n_1, n_2, h_1, h_2 are replaced by n_3, n_4, h_3, h_4. Making these substitutions in the above, we have

$$(5)\quad h = \tfrac{1}{2} n_1 n_2 n_3 n_4 (n_1 n_2 n_3 n_4 - n_1 - n_2 - n_3 - n_4 + 3) \\ + n_2 n_3 n_4 h_1 + n_3 n_4 n_1 h_2 + n_4 n_1 n_2 h_3 + n_1 n_2 n_3 h_4.$$

Without going through any further details we give at once the following formula, which can be easily verified, for the number of apparent double points on a curve C^N, where $N = n_1 n_2 \cdots n_q$:

$$(6)\qquad h = \tfrac{1}{2} n_1 n_2 \cdots n_q (n_1 n_2 \cdots n_q - \sum n_i + q - 1) \\ + n_1 n_2 \cdots n_q \sum^{q} h_i / n_i .$$

If $q = r - 1$, we have, from (2), $r_1 = r_2 = \cdots = r_{r-1} = r - 1$. Then the curve C^N is the intersection of $r - 1$ hypersurfaces. In this case, $h_1 = h_2 = \cdots = h_{r-1} = 0$ as a plane section of a hypersurface cannot have apparent double points. Then (6) is reduced to (1).

As an illustration, let C^9 be the intersection of a V^3_3 and a V'^3_3 in S_5. Since an S_3 in S_5 meets V^3_3 and V'^3_3 each in a twisted cubic curve, we have $h_1 = h_2 = 1$. We may use (3) or we may use (6) for $q = 2$. Putting $n_1 = n_2 = 3$, we have $h = 24$ as the number of apparent double points on the curve C^9.

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