GENERALIZATION OF A THEOREM OF KRONECKER

BY B. L. VAN DER WAERDEN

EXTRACT FROM A LETTER TO J. F. RITT

Your theorem on algebraic dependence* is, in the very special case in which all $\alpha_i, \beta_i, \gamma_i$ are powers of y, contained in a theorem of Kronecker.† Emmy Noether communicated to me some years ago a proof of Kronecker's theorem which can be extended as below, to your more general case. This proof is simpler than yours, and gives more information. The products $b_i c_j$ are not only algebraic, but *integral* algebraic.

HYPOTHESIS. *Let $\beta_1, \cdots, \beta_r; \gamma_1, \cdots, \gamma_s$ be two systems of linearly independent analytic functions of y. Let $b_1, \cdots, b_r; c_1, \cdots, c_s$ be indeterminates. Let*

$$(1)\ (b_1\beta_1 + \cdots + b_r\beta_r)(c_1\gamma_1 + \cdots + c_s\gamma_s) = (a_1\alpha_1 + \cdots + a_n\alpha_n),$$

where the α's are a linearly independent‡ set of products $\beta_i\gamma_j$ in terms of which all such products are expressible, and where the a's are linear combinations of the products b_ic_j.

CONCLUSION. *Every b_ic_j satisfies an equation of the form*

$$(2)\qquad Z^t + A_1Z^{t-1} + \cdots + A_t = 0,$$

with every A_k a homogeneous form of the kth degree in $a_1, \cdots, a_n$, with constant coefficients.

PROOF. If the expressions $a_1, \cdots, a_n$ are all zero for special values $b_1', \cdots, b_r'; c_1', \cdots, c_s'$ of the indeterminates b_i, c_j, it follows from (1) that

$$(3)\qquad (b_1'\beta_1 + \cdots + b_r'\beta_r)(c_1'\gamma_1 + \cdots + c_s'\gamma_s) = 0.$$

But if a product of analytic functions vanishes identically, one of the factors vanishes identically. If the first factor in (3) is zero, every b_i' is zero; if the second factor vanishes, then every c_i' does. In any case every product $b_i'c_j'$ vanishes. This means

---

* *On a certain ring of functions of two variables*, Transactions of this Society, vol. 32 (1930), p. 155.

† Berliner Sitzungsberichte, vol. 37 (1883), p. 957. (NOTE by J. F. Ritt: This relationship was known to me, but I was not in possession of the elegant methods of proof which Professor van der Waerden uses below.)

‡ With respect to all constants.

that every zero* of the ideal $(a_1, \cdots, a_n)$ is also a zero of the ideal

$$\theta = (\cdots, b_i c_j, \cdots) = (b_1, \cdots, b_r)(c_1, \cdots, c_s).$$

It follows from a well known theorem of Hilbert† that, for some positive integer t; $\theta^t \equiv 0, (a_1, \cdots, a_n)$, or, if one designates the products $b_i c_j$, in any order, by $d_1, \cdots, d_{rs}$,

$$d_{i_1} d_{i_2} \cdots d_{i_t} = \sum e_{i_1 \cdots i_t, k} a_k. \tag{4}$$

As the a's are linear combinations of the d's, the coefficients e in (4) may be taken‡ as homogeneous forms of degree $t-1$ in the d's. If the power products of the d's of degree $t-1$, written in any order, are designated by $g_1, \cdots, g_k$, then (4) may be written in the form $d_i g_j = \sum g_l a'_{ijl}$, where the a'_{ijl} are linear combinations of the a's. Elimination of the g's gives

$$\begin{vmatrix} d_i - a_{i11} & -a_{i12} & \cdots \\ -a'_{i21} & d_i - a'_{i22} & \cdots \\ \cdot\ \cdot & \cdot\ \cdot & \cdots \\ \cdot\ \cdot & \cdot\ \cdot & \cdots \end{vmatrix} = 0.$$

This is an algebraic equation for d_i of the form (2). The theorem is then proved.

If, now, the indeterminates b_i, c_j in (1) are replaced by other quantities, for instance functions of x, not necessarily analytic, the a's in the second member may become linearly independent. If they are all expressed in terms of the linearly dependent ones among them, the second member appears in "reduced form" (*On a certain ring*, etc., p. 157). Equation (2) holds identically and hence preserves its form when the b's and c's are replaced by other quantities, even if the a's are expressed in terms of the linearly independent ones among them. This proves your Theorem 1 (loc. cit., p. 156) with the additional information that the $b_i c_j$ are *integral* algebraic in the a's.

GRONINGEN, HOLLAND

---

* A set of values of the indeterminates for which every polynomial in the ideal vanishes.

† See Macaulay, *Modular Systems*, p. 46. (J. F. R.)

‡ The equation thus obtained may be written in the form of the Dedekind-Mertens "modulus-equation" $\theta^t = \theta^{t-1}\alpha$, $\alpha = (a_1, \cdots, a_n)$, which occurs in Dedekind's proof of Kronecker's theorem.