A CERTAIN MULTIPLE-PARAMETER EXPANSION*

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1. *Introduction.* C. C. Camp† has shown the convergence of the expansion of an arbitrary function in terms of the solutions of the systems of equations

$$X_1' + \left(\lambda a_1 - \sum_{j=2}^{n} \mu_j\right) X_1 = 0,$$

$$X_j' + (\lambda a_j + \mu_j) X_j = 0, \quad (j = 2, 3, \cdots, n),$$

where the a_j's are functions of x, with the boundary conditions

$$X_j(-\pi) = X_j(\pi), \qquad (j = 1, 2, \cdots, n).$$

In this paper it is intended to use a differential system in which the n parameters appear in each equation with a function $a_{jk}(x_j)$ multiplying each. The boundary conditions will also be more general.

2. *Expansion and Convergence Proof.* The system of equations

$$(1)\quad \begin{aligned} &X_j' + \left(\sum_{k=1}^{j} \lambda_k a_{jk}(x_j) - \sum_{k=j+1}^{n} \lambda_k a_{jk}(x_j)\right) X_j = 0, \\ &X_n' + \sum_{k=1}^{n} \lambda_k a_{nk}(x_n) X_n = 0, \qquad (j = 1, 2, \cdots, n-1), \end{aligned}$$

where the $a_{jk}(x_j)$, $(-a \leqq x_j \leqq b)$, are either positive functions or identically zero, with the boundary conditions

$$(2)\quad X_j(-a) = \gamma_j X_j(b), \quad (\gamma_j > 0), \quad (j = 1, 2, \cdots, n),$$

has the solutions

$$(3)\quad X_j = \exp\left\{-\sum_{1}^{j} \lambda_k A_{jk}(x_j) + \sum_{j+1}^{n}{}' \lambda_k A_{jk}(x_j)\right\}, \qquad (j = 1, 2, \cdots, n),$$

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† American Journal of Mathematics, vol. 50 (1928), p. 259.

where the symbol $\sum_{j+1}^{n}{}'$ is defined to be zero when $j=n$, and $\int a_{jk}(x_j)dx_j = A_{jk}(x_j)$.

From the boundary conditions we obtain the expression

$$\exp\left\{-\sum_{1}^{j}\lambda_k A_{jk}(-a) + \sum_{j+1}^{n}{}'\lambda_k A_{jk}(-a)\right\}$$
$$= \gamma_j \exp\left\{-\sum_{1}^{j}\lambda_k A_{jk}(b) + \sum_{j+1}^{n}{}'\lambda_k A_{jk}(b)\right\},$$

or

$$(4)\ \exp\left\{\left(\sum_{1}^{j}\lambda_k A_{jk} - \sum_{j+1}^{n}{}'\lambda_k A_{jk}\right)(a+b)\right\} = \gamma_j \exp\left\{2\pi m_j i\right\},$$
$$(m_j = 0,\ \pm 1,\ \pm 2, \cdots),$$

where

$$\int_{-a}^{b} a_{jk}(x_j)dx_j = (a+b)A_{jk}.$$

Whence, the principal parameter values of

$$(5)\qquad \nu_j \equiv \sum_{1}^{j}\lambda_k A_{jk} - \sum_{j+1}^{n}{}'\lambda_k A_{jk}$$

are $\nu_j^* = (\log \gamma_j + 2\pi m_j i)/(a+b)$, the new parameters ν_j being introduced for simplicity. It may be noticed that the ν^*'s lie on lines parallel to the imaginary axis at a distance $\log \gamma_j/(a+b)$.

The system (5) may be solved for λ_k giving

$$(6)\qquad \lambda_k = \sum_{j=1}^{n}(-1)^{j+k}\nu_j D(A_{jk})/D(A),$$

where it is assumed that $D(A)\neq 0$, $D(A)$ being the determinant of (5), and $D(A_{jk})$ the minor of A_{jk} in $D(A)$. $D(a)$ and $D(a_{jk})$ are similar determinants, but with each A_{jk} replaced by the corresponding $a_{jk}(x_j)$.

The system adjoint to (1), (2) is

$$(1')\qquad -Y_j' + \left(\sum_{k=1}^{j}\lambda_k a_{jk}(x_j) - \sum_{k=j+1}^{n}{}'\lambda_k a_{jk}(x_j)\right)Y_j = 0,$$

$$(2')\qquad \gamma_j Y_j(-a) = Y_j(b),$$

which has the same characteristic values λ_k^*, ν_j^* as (1), (2).

Let one asterisk designate any one set of parameter values and two asterisks any other set. Consider the two sets of equations, (1) using one asterisk, and (1′) using two. Multiply the first set by Y_j^{**} and the second by $-X_j^*$, add, and integrate from $-a$ to b, obtaining

$$\int_{-a}^{b}(X_j^{*\prime}Y_j^{**}+X_j^*Y_j^{**\prime})dx_j$$
$$+\sum_{k=1}^{j}(\lambda_k^*-\lambda_k^{**})\int_{-a}^{b}a_{jk}(x_j)X_j^*Y_j^{**}dx_j$$
$$-\sum_{k=j+1}^{n}{}'(\lambda_k^*-\lambda_k^{**})\int_{-a}^{b}a_{jk}(x_j)X_j^*Y_j^{**}dx_j=0.$$

Since the first integral vanishes at the limits, and since $\lambda_k^*\neq\lambda_k^{**}$ for at least one value of k, the determinant of the coefficients of the quantities involving λ_k must vanish, giving the conjugacy condition,

$$\text{(7)}\qquad \int_{-a}^{b}{}^{(n)}\; D(a)\prod_{j=1}^{n}X_j^*Y_j^{**}dx_j=0,$$

where the symbol (n) means an n-fold integral. If the characteristic values in X_j and Y_j all happen to coincide the above integral becomes, since $\prod_{n=1}^{i}X_j^*Y_j^*=1$,

$$\text{(8)}\qquad \int_{-a}^{b}{}^{(n)}\; D(a)\prod_{j=1}^{n}dx_j=(a+b)^nD(A).$$

Assuming the expansion

$$f(x)=\sum_{m_j=-\infty}^{+\infty}C_{m_j}\prod_{1}^{n}X_j^*,\quad (x\equiv x_1,x_2,\cdots,x_n),$$

where $f(x)$ is made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each variable, multiply both sides by $D(a)\prod_1^n Y_j^{**}$ and integrate n times from $-a$ to b. On the right, all the terms where the characteristic numbers are different, vanish, and the one where they coincide gives $C_{m_j}(a+b)^nD(A)$. The constant C_{m_j} may thus be determined and the expansion becomes

$$\text{(9)}\quad f(x)=\sum_{m_j=-\infty}^{+\infty}\int_{-a}^{b}{}^{(n)}\; f(u)D(a)\prod_{1}^{n}X_j^*(x)Y_j^*(u)du_j/\left[(a+b)^nD(A)\right].$$

Green's function for this system is

$$(10)\quad G \equiv \begin{cases} \prod_1^n [X_j^*(x)Y_j^*(u)/(1-\gamma_j e^{-\nu_j(a+b)})]/D(A), & (u_j < x_j), \\ \prod_1^n [X_j^*(x)Y_j^*(u)/(e^{\nu_j(a+b)}/\gamma_j - 1)]/D(A), & (u_j > x_j), \end{cases}$$

the residue of which with respect to ν_j is

$$\prod_1^n X_j^*(x)\,Y_j^*(u)/[D(A)(a+b)^n].$$

Hence the residue of

$$\int_a^{b\,(n)} f(u)D(a)G\prod_1^n du_j$$

is

$$\int_{-a}^{b\,(n)} f(u)D(a)\ \prod_1^n X_j^*(x)Y_j^*(u)du_j/[D(A)(a+b)^n]$$

which is one of the terms of the expansion.

To show that the expansion converges, an extension† of Birkhoff's contour integral method will be used. In the evaluation of the contour integrals the following lemma will be useful.

LEMMA.

$$\lim_{m\to\infty}\frac{1}{2\pi i}\int_{\Gamma_m}\frac{|e^{-h(a+b)\nu}|\,d\nu}{(1-\gamma e^{-(a+b)\nu})\nu} = \begin{cases} 0, & (0 < h < 1), \\ \tfrac{1}{2}, & (h = 0), \\ -\tfrac{1}{2}, & (h = 1), \end{cases}$$

where Γ_m represents any contour which encloses the $(2m+1)$ poles of the integrand, one of which is at the origin with the rest equally spaced on a line parallel to the imaginary axis.

One may use circles with centers at the origin, expanding in such a way that they always pass midway between successive poles, or rectangles of fixed width that include the line of poles and the origin, with ends that recede indefinitely and pass midway between the poles as before.

The contour integral to be evaluated in the convergence proof has 2^n pieces, one of which has the form

† Camp, loc. cit.

$$(11)\qquad I_{-a}^{x} = \lim_{m_j\to\infty} \overset{(n)}{\int_{\Gamma_{m_j}}} \overset{(n)}{\int_{-a}^{x}}$$

$$\frac{f(u)D(a)\prod_1^n \exp\left\{\int_{x_j}^{u_j}\left[\sum_1^j \lambda_k a_{jk}(x_j) - \sum_{j+1}^{n}{}' \lambda_k a_{jk}(x_j)\right]dx_j\right\} du_j dv_j}{(2\pi i)^n D(A)\prod_1^n (1-\gamma_j e^{-v_j(a+b)})},$$

where the contours Γ_{m_j} are those used in the lemma. The "u" integrals will be evaluated first. $D(a)$ is replaced by the identical expression

$$(1/\lambda_1)\sum_{j=1}^{n}(-1)^{j+1}D(a_{j1})\left[\sum_{k=1}^{j}\lambda_k a_{jk}(x_j) - \sum_{k=j+1}^{n}{}' \lambda_k a_{jk}(x_j)\right]$$

and each term of the summation with the exponential factor containing the corresponding variable is integrated by parts giving

$$(12)\quad \left\{\sum_{j=1}^{n}(-1)^{j+1} \overset{(n-1)}{\int_{-a}^{x^j}}\left[f(x_j, u^j)\right.\right.$$

$$\left. - f(-a, u^j)\exp\left\{\int_{x_j}^{-a}\left[\sum_1^j \lambda_k a_{jk}(x_j) - \sum_{j+1}^{n}{}' \lambda_k a_{jk}(x_j)\right]dx_j\right\}\right]$$

$$\cdot D(a_{j1})\prod_{l=1}^{n}{}^{j}\exp\left\{\int_{x_l}^{u_l}\left[\sum_{k=1}^{l}\lambda_k a_{lk}(x_l) - \sum_{k=l+1}^{n}{}'\lambda_k a_{lk}(x_l)\right]dx_l\right\} du_l$$

$$- \overset{(n)}{\int_{-a}^{x}}\left[\sum_{j=1}^{n}(-1)^{j+1}\frac{\partial f(u)}{\partial u_j}D(a_{j1})\right]\prod_1^n \exp\left\{\int_{x_j}^{u_j}\left[\sum_1^j \lambda_k a_{jk}(x_j)\right.\right.$$

$$\left.\left.\left. - \sum_{j+1}^{n}{}'\lambda_k a_{jk}(x_j)\right]dx_j\right\}du_j\right\}\Big/ [D(A)\lambda_1]$$

where the superscript j means that the variable with the corresponding subscripts is missing, that is, $x^j \equiv x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n$. Thus $f(x_2, u^2) \equiv f(u_1, x_2, u_3, \cdots, u_n)$. A superscript with a $\sum$ or $\prod$ sign has a similar meaning.

To continue the integration of the terms involving $f(x_j, u^j)$, the first minors, $D(a_{j1})$, are developed according to rows in terms of second minors, thus making $n-1$ terms in place of each. As each row in $D(a)$ contains only the variables x_j, each of the

$n-1$ terms obtained from $D(a_{j1})$ consists of $a_{j'k}(x_{j'})$ and a second minor independent of $x_{j'}$. In this integration j' is different from the j in the original integration. In the exponential factor of $a_{j'k}(x_{j'})$ that involves $x_{j'}$, replace λ_k by its value obtained by solving the jth equation of (5). The exponent will be

$$\int_{x_{j'}}^{u_{j'}} \Bigg[\nu_{j'} a_{j'k}(x_{j'}) dx_{j'} \\ - \left(\sum_{q=1}^{j'} - \sum_{q=j'+1}^{n}{}' \right)^{k} \lambda_q \begin{vmatrix} a_{j'k}(x_{j'}) & a_{j'q}(x_{j'}) \\ A_{j'k} & A_{j'q} \end{vmatrix} dx_{j'} \Bigg] \Big/ A_{j'k} .$$

If in an integration by parts,

$$a_{j'k}(x_{j'}) \ \exp \left\{ \int_{x_{j'}}^{u_{j'}} \nu_{j'} a_{j'k}(x_{j'}) dx_{j'} / A_{j'k} \right\} du_{j'}$$

is taken as the "dv" factor, the upper limit term of the integrated part will be $f(x_j, x_{j'}, u^{j,j'}) A_{j'k}/\nu_{j'}$, j' being any j different from the former j, with a similar expression for each of the $n-1$ terms. Continuing this process by developing the second minors by rows and third minors, integrating by parts again with respect to a different x_j, and repeating until all n integrations have been performed, we find that the expression $f(x)D(A_{j1})/\prod_{l=1}^{n}\nu_l$ results from each of the terms involving $f(x_j, u^j)D(a_{j1})$. Then by adding the n similar terms obtained from the whole summation, and dividing by the factor $\lambda_1 D(A)$, we obtain the expression $f(x)/\prod_1^n \nu_j$.

The n-fold contour integration of this expression by means of the lemma gives the result $f(x)/2^n$. The other 2^n-1 "I" integrals give similar results, the combination of which form the so-called mean value of $f(x)$. If the lower limit terms and other integrals that arise in the various integrations by parts can be shown to contribute zero in the limit, the convergence of the expansion may be considered proved.

The integrals of the lower limit terms of (12) may be evaluated by comparing them with a simpler series obtained by the following method. First, replace each $f(-a, u^j)$ by a common upper bound, and expand each $D(a_{j1})$ by any row and second minors. Next replace each λ_k in the exponential factor of $a_{j'k}(x_{j'})$ that involves $x_{j'}$ by its value obtained from the jth equation of (5) as in the preceding integration. Then take absolute values and remove the exponential factors that still contain λ's from the integrals by the first mean value theorem. The integration with respect to $x_{j'}$ may now be performed. Repeat the process by expanding the second minors by rows into third minors, and so on until $n-1$ integrations have been performed. To carry out the contour integration put all resulting terms over the same denominator $\prod_1^n \nu_j$, and remove from each term the exponential factors involving λ's and the factors $\nu_j/[D(A)\lambda_1]$ all of which are bounded on the rectangular contours mentioned in the lemma. The remaining exponentials are of the form

$$\exp\left\{-\nu_j\int_{-a}^{x_j} a_{jk}(x_j)dx_j/A_{jk}\right\} = \exp\left\{-\nu_j h(a+b)\right\},$$

$$(0 < h < 1),$$

and the contour integrals accordingly contribute zero in the limit. All the lower limit terms that arise in the other integrations by parts may be treated similarly.

If $f(x)$ is only piecewise (stückweise) continuous, at each point of discontinuity certain terms will occur in the integration by parts which contribute zero to the final result after a contour integration similar to that used for the lower limit terms.

In the various integrations by parts, the unintegrated terms will have the form

$$-\int_{-a}^{x_{j'}}\left[\frac{\partial f(x_j, u^j)}{\partial u_j}\right.$$

$$\left.-\left(\sum_{q=1}^{j'}-\sum_{q=j'+1}^{n}{}'\right)^k\lambda_q\begin{vmatrix} a_{j'k}(x_{j'}) & a_{j'q}(x_{j'})\\ A_{j'k} & A_{j'q}\end{vmatrix}\Big/A_{j'k}\right]\frac{A_{j'k}}{\nu_{j'}}$$

$$\cdot\exp\left\{\int_{x_{j'}}^{u_{j'}}\left[\nu_{j'}a_{j'k}(x_{j'})\right.\right.$$

$$\left.\left.-\left(\sum_{q=1}^{j'}-\sum_{q=j'+1}^{n}{}'\right)^k\lambda_q\begin{vmatrix} a_{j'k}(x_{j'}) & a_{j'q}(x_{j'})\\ A_{j'k} & A_{j'q}\end{vmatrix}\right]dx_{j'}/A_{j'k}\right\}du_{j'}$$

with the remaining exponentials and a second minor as factors in the $(n-1)$-fold integral. To integrate, replace the factor in the bracket by an upper bound, take absolute values and remove the exponential by the first mean value theorem, obtaining

$$- A_{j'k} \exp \left\{ \int_{x_{j'}}^{\eta} (\cdots) dx_{j'} \right\} (a + x_{j'})/\nu_{j'}, \qquad (u_{j'} < \eta < x_{j'}),$$

as a result. For the other $n-2$ integrations expand the second minors by rows and third minors, remove exponential factors containing λ's by the first mean value theorem, and proceed as before. The contour integrals will then approach zero in the limit.

To determine the contribution of the last integral in (12), replace each $\partial f(u)/\partial u_j$ by a common upper bound, expand each $D(a_{j1})$ by rows and second minors, take absolute values and proceed as for the lower limit terms for $n-1$ integrations. To make the nth integration, remove the exponential by the first mean value theorem and integrate directly Then, if we remove bounded factors, the contour integration follows as before and contributes zero in the limit. We have thus proved the following theorem.

THEOREM. *Let $f(x)$ be made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each argument in the region $-a \leqq x_j \leqq b$, while each $a_{jk}(x_j)$ is integrable and either positive or identically zero over the region. Then the expansion $\sum_{m_j} C_{m_j} \prod_1^n X_j^*$, $(m_j = 0, \pm 1, \pm 2, \cdots)$, where X_j^* is a characteristic solution of the jth equation of* (1), (2), *and where*

$$C_{m_j} = \int_{-a}^{b} {}^{(n)} f(u) D(a) \prod_1^n Y_j^*(u) du_j / [(a + b)^n D(A)],$$

converges to the so-called mean value of $f(x)$ at any interior point of the region.

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