AN EXISTENCE THEOREM FOR CHARACTERISTIC CONSTANTS OF KERNELS OF POSITIVE TYPE*

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We define a kernel $K(s, t)$ to be of positive type with respect to a set of functions $H(s)$ if

$$(1)\qquad \int_a^b \int_a^b K(s, t)h(s)h(t)dsdt \geqq 0$$

for every function $h(s)$ of the set $H(s)$.

Let the real kernel $K(s, t)$ be developable in a series of real normalized orthogonal functions $\phi_i(s)$, so that

$$(2)\qquad K(s, t) = \sum_{i,j=1}^{\infty} a_{ij}\phi_i(s)\phi_j(t),$$

where

$$(3)\qquad a_{ij} = \int_a^b \int_a^b K(s, t)\phi_i(s)\phi_j(t)dsdt,$$

and not all the a_{ij} are zero. We shall prove the following theorem.

THEOREM 1. *If $K(s, t)$ is of positive type with respect to the set of all functions of the form $c_\alpha\phi_\alpha(s)+c_\beta\phi_\beta(s)$, where the c's are real constants, then no coefficient a_{kk} is negative, and no a_{kk} is zero unless $a_{kj}+a_{jk}=0$ for every j.*

Suppose, for some subscript k, we have $a_{kk}<0$. Let $h(s)=\phi_k(s)$ in (1). Then we have, since the functions $\phi_j(s)$ form a normalized orthogonal set,

$$(4)\qquad \begin{aligned} &\int_a^b \int_a^b K(s, t)\phi_k(s)\phi_k(t)dsdt \\ &= \int_a^b \int_a^b \sum_{i,j} a_{ij}\phi_i(s)\phi_j(t)\phi_k(s)\phi_k(t)dsdt \\ &= \int_a^b \sum_i a_{ik}\phi_i(s)\phi_k(s)ds = a_{kk} < 0, \end{aligned}$$

and we see that condition (1) is not satisfied.

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Suppose now $a_{kk}=0$, and $a_{kj}+a_{jk}\neq 0$ for some j. Let us set $h(s)=c_k\phi_k(s)+c_j\phi_j(s)$ in condition (1), where the c's are arbitrary real constants. We have

$$\int_a^b \int_a^b K(s, t)\left[c_k\phi_k(s) + c_j\phi_j(s)\right]\left[c_k\phi_k(t) + c_j\phi_j(t)\right]dsdt$$

$$= \int_a^b \int_a^b \left[\sum_{\alpha,\beta} a_{\alpha\beta}\phi_\alpha(s)\phi_\beta(t) \right]\left[c_k\phi_k(s) + c_j\phi_j(s)\right]\left[c_k\phi_k(t)\right.$$

$$\text{(5)} \qquad \left. + c_j\phi_j(t)\right]dsdt$$

$$= \int_a^b \left[\sum_\alpha c_k a_{\alpha k}\phi_\alpha(s) + \sum_\alpha c_j a_{\alpha j}\phi_\alpha(s) \right]\left[c_k\phi_k(s) + c_j\phi_j(s)\right]ds$$

$$= c_k^2 a_{kk} + c_j c_k a_{kj} + c_k c_j a_{jk} + c_j^2 a_{jj}.$$

Since $a_{kk}=0$, the right member of (5) reduces to

$$\text{(6)} \qquad c_j^2 a_{jj} + c_j c_k(a_{kj} + a_{jk}).$$

Since $a_{kj}+a_{jk}\neq 0$, we can choose c_j and c_k so that the quantity (6) is negative. This shows that a function $h(s)$ can be found for which condition (1) is not satisfied, which completes the proof of our theorem.

We now make the following additional assumptions for the kernel $K(s, t)$:

(i) $K(s, t)$ satisfies the continuity conditions required for applicability of the Fredholm theory.*

(ii) The series obtained by termwise integration of $\sum_{i,j=1}^{\infty} a_{ij}\phi_i(s)\phi_j(s)$ is convergent and represents $\int_a^b K(s, s)ds$.

(iii) The Fredholm determinant $D(\lambda)$ of $K(s, t)$ is of genus zero.

A sufficient condition for (iii) is that

$$\left| \frac{K(s, t_1) - K(s, t)}{t_1 - t} \right| < M$$

for some constant M, independent of s, t, and t_1.†

From (i) we have the well known development

* See, for example, Goursat, *Cours d'Analyse*, vol. 3, p. 342.

† See Fredholm, Acta Mathematica, vol. 27, p. 368.

$$(7)\ D(\lambda) = 1 - \lambda \int_a^b K(s, s)ds + \cdots$$

$$+ \frac{(-\lambda)^p}{p!} \int_a^b \cdots \int_a^b K\begin{pmatrix} s_1 \cdots s_p \\ s_1 \cdots s_p \end{pmatrix} ds_1 \cdots ds_p + \cdots ,$$

and from (iii)*

$$(8) \qquad D(\lambda) = \prod_{i=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_i}\right) = 1 - \lambda\left(\sum \frac{1}{\lambda_i}\right) + \cdots ,$$

where the λ_i are the characteristic constants of the kernel.

Equating coefficients of λ in (7) and (8) and applying assumption (ii), we have

$$(9) \qquad \int_a^b K(s, s)ds = \sum \frac{1}{\lambda_i} = \sum a_{ii}.$$

It is clear that, under our present assumptions, $K(s, t)$ must have at least one finite characteristic constant if $\sum a_{ii} \neq 0$.

Now $\sum a_{ii}$ cannot vanish unless either some a_{kk} is negative or every a_{kk} is zero. From Theorem 1, $K(s, t)$ cannot be of positive type with respect to the set of all functions of the form $c_\alpha \phi_\alpha(s) + c_\beta \phi_\beta(s)$ if any a_{kk} is negative, nor if any a_{kk} vanishes unless $a_{kj} + a_{jk} = 0$ for every j. If every a_{kj} and a_{jk} vanishes, we have the trivial case where $K(s, t)$ vanishes identically, which has been excluded. But if every a_{kk} is zero and for every k and j, $a_{kj} = -a_{jk}$, $K(s, t)$ is skew-symmetric: $K(s, t) = -K(t, s)$, and hence has at least one characteristic constant.†

We have therefore proved the following theorem.

THEOREM 2. *If a given kernel* $K(s, t)$, *developable in a series* $\sum a_{ij}\phi_i(s)\phi_j(t)$ *of normalized orthogonal functions, is of positive type with respect to the set of all functions of the form* $c_\alpha\phi_\alpha(s) + c_\beta\phi_\beta(s)$, *and satisfies the conditions* (i), (ii), *and* (iii), *it has at least one finite characteristic constant.*

It will be noted that the conditions (i), (ii), and (iii) all refer to continuity properties rather than to symmetry.

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* See, for example, Goursat, *Cours d'Analyse*, vol. 2, pp. 152–153; vol. 3, pp. 425–426.

† See, for example, Goursat, *Cours d'Analyse*, vol. 3, p. 468.