AN EXISTENCE THEOREM FOR CHARACTERISTIC
CONSTANTS OF KERNELS OF POSITIVE TYPE*

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We define a kernel $K(s, t)$ to be of positive type with respect to a set of functions $H(s)$ if
\[
\int_a^b \int_a^b K(s, t) h(s) h(t) ds dt \geq 0
\]
for every function $h(s)$ of the set $H(s)$.

Let the real kernel $K(s, t)$ be developable in a series of real normalized orthogonal functions $\phi_i(s)$, so that
\[
K(s, t) = \sum_{i, j=1}^n a_{ij} \phi_i(s) \phi_j(t),
\]
where
\[
a_{ij} = \int_a^b \int_a^b K(s, t) \phi_i(s) \phi_j(t) ds dt,
\]
and not all the $a_{ij}$ are zero. We shall prove the following theorem.

THEOREM 1. If $K(s, t)$ is of positive type with respect to the set of all functions of the form $c_1 \phi_1(s) + c_2 \phi_2(s)$, where the $c$'s are real constants, then no coefficient $a_{kk}$ is negative, and no $a_{kk}$ is zero unless $a_{kj} + a_{jk} = 0$ for every $j$.

Suppose, for some subscript $k$, we have $a_{kk} < 0$. Let $h(s) = \phi_k(s)$ in (1). Then we have, since the functions $\phi_j(s)$ form a normalized orthogonal set,
\[
\int_a^b \int_a^b K(s, t) \phi_k(s) \phi_k(t) ds dt
\]
\[
= \int_a^b \int_a^b \sum_{i, j} a_{ij} \phi_i(s) \phi_j(t) \phi_k(s) \phi_k(t) ds dt
\]
\[
= \int_a^b \sum_i a_{ik} \phi_i(s) \phi_k(s) ds = a_{kk} < 0,
\]
and we see that condition (1) is not satisfied.

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Suppose now \( a_{kk} = 0 \), and \( a_{kj} + a_{jk} \neq 0 \) for some \( j \). Let us set \( h(s) = c_k \phi_k(s) + c_j \phi_j(s) \) in condition (1), where the \( c \)'s are arbitrary real constants. We have

\[
\int_a^b \int_a^b K(s, t) [c_k \phi_k(s) + c_j \phi_j(s)] [c_k \phi_k(t) + c_j \phi_j(t)] dsdt
\]

\[
= \int_a^b \int_a^b \left[ \sum_{a,j} a_{aj} \phi_a(s) \phi_j(t) \right] [c_k \phi_k(s) + c_j \phi_j(s)] [c_k \phi_k(t) + c_j \phi_j(t)] dsdt
\]

\[
= \int_a^b \left[ \sum_a c_k a_{kk} \phi_a(s) + \sum_j c_j a_{kj} \phi_j(s) \right] [c_k \phi_k(s) + c_j \phi_j(s)] ds
\]

\[
= c_k^2 a_{kk} + c_j c_k a_{kj} + c_k c_j a_{jk} + c_j^2 a_{jj}.
\]

Since \( a_{kk} = 0 \), the right member of (5) reduces to

\[
c_j^2 a_{jj} + c_j c_k (a_{kj} + a_{jk}).
\]

Since \( a_{kj} + a_{jk} \neq 0 \), we can choose \( c_j \) and \( c_k \) so that the quantity (6) is negative. This shows that a function \( h(s) \) can be found for which condition (1) is not satisfied, which completes the proof of our theorem.

We now make the following additional assumptions for the kernel \( K(s, t) \):

(i) \( K(s, t) \) satisfies the continuity conditions required for applicability of the Fredholm theory.\(^*\)

(ii) The series obtained by termwise integration of \( \sum_{i,j=1}^n a_{ij} \phi_i(s) \phi_j(s) \) is convergent and represents \( f_a^b K(s, s) ds \).

(iii) The Fredholm determinant \( D(\lambda) \) of \( K(s, t) \) is of genus zero.

A sufficient condition for (iii) is that

\[
\left| \frac{K(s, t_1) - K(s, t)}{t_1 - t} \right| < M
\]

for some constant \( M \), independent of \( s \), \( t \), and \( t_1 \).\(^†\)

From (i) we have the well known development

\* See, for example, Goursat, Cours d'Analyse, vol. 3, p. 342.

\[ D(\lambda) = 1 - \lambda \int_a^b K(s, s)ds + \cdots \]
\[ + \frac{(-\lambda)^p}{p!} \int_a^b \cdots \int_a^b K(s_1 \cdots s_p)ds_1 \cdots ds_p + \cdots, \]
and from (iii)*
\[ D(\lambda) = \prod_{i=1}^\infty \left( 1 - \frac{\lambda}{\lambda_i} \right) = 1 - \lambda \left( \sum \frac{1}{\lambda_i} \right) + \cdots, \]
where the \( \lambda_i \) are the characteristic constants of the kernel.

Equating coefficients of \( \lambda \) in (7) and (8) and applying assumption (ii), we have
\[ \int_a^b K(s, s)ds = \sum \frac{1}{\lambda_i} = \sum a_{ii}. \]

It is clear that, under our present assumptions, \( K(s, t) \) must have at least one finite characteristic constant if \( \sum a_{ii} \neq 0 \).

Now \( \sum a_{ii} \) cannot vanish unless either some \( a_{kk} \) is negative or every \( a_{kk} \) is zero. From Theorem 1, \( K(s, t) \) cannot be of positive type with respect to the set of all functions of the form \( c_\alpha \phi_\alpha(s) + c_\beta \phi_\beta(s) \) if any \( a_{kk} \) is negative, nor if any \( a_{kk} \) vanishes unless \( a_{kj} + a_{jk} = 0 \) for every \( j \). If every \( a_{kj} \) and \( a_{jk} \) vanishes, we have the trivial case where \( K(s, t) \) vanishes identically, which has been excluded. But if every \( a_{kk} \) is zero and for every \( k \) and \( j, a_{kj} = -a_{jk} \), \( K(s, t) \) is skew-symmetric: \( K(s, t) = -K(t, s) \), and hence has at least one characteristic constant.†

We have therefore proved the following theorem.

**Theorem 2.** If a given kernel \( K(s, t) \), developable in a series \( \sum a_{ij} \phi_i(s)\phi_j(t) \) of normalized orthogonal functions, is of positive type with respect to the set of all functions of the form \( c_\alpha \phi_\alpha(s) + c_\beta \phi_\beta(s) \), and satisfies the conditions (i), (ii), and (iii), it has at least one finite characteristic constant.

It will be noted that the conditions (i), (ii), and (iii) all refer to continuity properties rather than to symmetry.

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† See, for example, Goursat, *Cours d’Analyse*, vol. 3, p. 468.