SOME THEOREMS ON PLANE CURVES

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In applying Abel's theorem to hyperelliptic integrals, we are interested in the intersections of certain curves with a curve \( C \) of the type \( y^2 = f(x) \), where \( f(x) \) is a polynomial. The functions used in the following are all polynomials of degree indicated by their subscripts. If \( f_n(x) = f_k(x)f_{n-k}(x) \) we may without any loss of generality assume that \( n \geq k \geq n/2 \) and this assumption will be made throughout.

**LEMMA.** If \( C \) is the curve \( y^2 = f_n(x) \equiv f_k(x)f_{n-k}(x) \), \( c_1 \) the curve \( y = f_k(x) \) and \( c_2 \) the curve \( y = f_{n-k}(x) \), then all the finite points of intersection of \( c_1 \) and \( c_2 \) are on \( C \), and the curve \( S \) whose equation is \( y = \left[ f_k(x) + f_{n-k}(x) \right]/2 \) is tangent to \( C \) at each of these \( k \) points.

Suppose \((\alpha, \beta)\) is any one of the \( k \) points of intersection of \( c_1 \) and \( c_2 \); then \( \beta = f_k(\alpha) \) and \( \beta = f_{n-k}(\alpha) \) and therefore \( \beta^2 = f_k(\alpha)f_{n-k}(\alpha) = f_n(\alpha) \), that is \((\alpha, \beta)\) is on \( C \). Obviously \( S \) passes through the \( k \) points of intersection of \( c_1 \) and \( c_2 \) and hence meets \( C \) in these \( k \) points. Eliminating \( y \) from the equations of \( S \) and \( C \) we get

\[
\left[ \frac{f_k(x) + f_{n-k}(x)}{2} \right]^2 - f_k(x)f_{n-k}(x) = \left[ \frac{f_k(x) - f_{n-k}(x)}{2} \right]^2 = 0
\]

as the equation giving the abscissas of the \( 2k \) points of intersection of \( S \) and \( C \). Since the left hand side of this equation is a perfect square each abscissa is counted twice, and therefore since, in \( S \), \( y \) is a one-valued function of \( x \), \( S \) is tangent to \( C \) at each of these \( k \) points.

As an immediate consequence of this lemma we have the following result.

**Theorem 1.** If \( C \) is the curve \( y^2 = \phi_n(x) \), where \( \phi_n(e_i) = 0 \), \((i = 1, \ldots, n)\), and \((\alpha, \beta), (\beta \neq 0)\), is a point on \( C \), and \( c_1 \) is the curve of the form \( y = \phi_k(x) \) determined by \((\alpha, \beta)\) and any \( k \) of the points \((e_i, 0)\), and \( c_2 \) is the curve of the form \( y = \phi_{n-k}(x) \) determined by \((\alpha, \beta)\) and the remaining \( n-k \) of the points \((e_i, 0)\), then \( c_1 \) and
$c_2$ have all their $k$ points of intersection* on $C$, and the curve $S$ whose
equation is $y = \left[\phi_k(x) + \phi_{n-k}(x)\right]/2$ is tangent to $C$ at each of these
$k$ points.

Since $\phi_n(x) = \phi_k(x)\phi_{n-k}(x)$ for $n+1$ values of $x$, we have
$\phi_n(x) = \phi_k(x)\phi_{n-k}(x)$ and the theorem follows from the lemma.

That all curves $S$ of the form $y = g_k(x)$ which are tangent to a
curve $C$ of the form $y^2 = g_n(x)$ at each of $k$ points can be obtained
by this process, is a consequence of the following theorem.

**Theorem 2.** If $(\alpha_i, \beta_i)$, $(i = 1, 2, \ldots, k)$, are $k$ points on the
curve $C$ whose equation is $y^2 = g_n(x)$ such that there exists a curve $S$
of the form $y = g_k(x)$ which is tangent to $C$ at each of these $k$ points,
and if the curve $c_1$ whose equation is $y = h_k(x)$ meets $C$ in the $k$
points $(\alpha_i, \beta_i)$ and the point $(e_\lambda, 0)$, where $e_\lambda$ is any zero of $g_n(x)$,
then $h_k(x)$ is a factor of $g_n(x)$.

Since $S$ is tangent to $C$ at each of the $k$ points, the equation
$g_k^2(x) - g_n(x) = 0$ has the roots $\alpha_1, \alpha_2, \ldots, \alpha_k$, each counted
twice, and since $c_1$ meets $S$ in the $k$ points $(\alpha_i, \beta_i)$, the equation
$g_k(x) - h_k(x) = 0$ has the roots $\alpha_1, \alpha_2, \ldots, \alpha_k$.

We have therefore

$$[g_k(x) - h_k(x)]^2 = \mu [g_k^2(x) - g_n(x)],$$

and hence

$$[g_k(e_\lambda) - h_k(e_\lambda)]^2 = \mu [g_k^2(e_\lambda) - g_n(e_\lambda)];$$

but $h_k(e_\lambda) = g_n(e_\lambda) = 0$, hence $\mu = 1$, and we have

$$g_k^2(x) - 2g_k(x)h_k(x) + h_k^2(x) \equiv g_k^2(x) - g_n(x),$$

or

$$g_n(x) \equiv h_k(x)[2g_k(x) - h_k(x)].$$

If $c_1$ is the curve $y = a_0 x^k + a_1 x^{k-1} + \cdots + a_{k-1} x + a_k$ deter-
mined by the $k$ points $(\alpha_i, \beta_i)$ and one of the $n$ points $(e_\lambda, 0)$, the
coefficient $a_0$ may be zero and the degree of the right hand side
less than $k$. For suppose we choose a particular one, say $e_1$, of

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* Only finite points of intersection are considered here. In certain special
cases when $n$ is even and $k = \frac{n}{2}$, $c_1$ and $c_2$ may coincide or they may have less than $k$
finite points of intersection. The lemma and Theorem 1 are still true for
these cases when finite points of intersection are considered.
the zeros of $g_n(x)$ and find that the expression on the right is of degree $k$; then it will have as zeros $k$ of the zeros of $g_n(x)$, say $e_1, e_2, \ldots, e_k$. Then the curve $y = b_0x^k + b_1x^{k-1} + \cdots + b_{k-1}x + b_k$ determined by the $k$ points $(\alpha_i, \beta_i)$ and one of the remaining points $(e_i, 0)$, say $(e_{k+1}, 0)$, will have its right hand side of degree $n - k$ at most. For suppose the right hand side of degree $m > n - k$; then it will have as zeros $m$ of the zeros of $g_n(x)$ and hence at least one of the $e_1, e_2, \ldots, e_k$ and therefore $a_0x^k + a_1x^{k-1} + \cdots + a_k = b_0x^k + b_1x^{k-1} + \cdots + b_k$ for at least $k+1$ values. But since $b_0x^k + b_1x^{k-1} + \cdots + b_k$ has at least one zero which is not a zero of $a_0x^k + a_1x^{k-1} + \cdots + a_k$ this is impossible. It follows as a consequence of Theorem 1 that the degree of the right hand side is either $k$ or $n - k$ depending on which zero of $g_n(x)$ is chosen for determining the curve $c_i$.

If in the above the degree of $h_k(x)$ is $k$, the degree of $2g_k(x) - h_k(x)$ will be $n - k$; if we denote the latter by $h_{n-k}(x)$, we shall have $g_k(x) = [h_k(x) + h_{n-k}(x)]/2$. That is, the curve $S$ is $y = [h_k(x) + h_{n-k}(x)]/2$, where the curve $y = h_k(x)$ is determined by some $k$ of the points $(e_i, 0)$ and one of the points $(\alpha_i, \beta_i)$, and the curve $y = h_{n-k}(x)$ is determined by the remaining $n-k$ of the points $(e_i, 0)$ and the same one of the points $(\alpha_i, \beta_i)$.

Thus far it has not been necessary to say anything about the nature of the zeros $e_1, e_2, \ldots, e_n$. When these zeros are distinct we have the following theorem.

**Theorem 3.** The number of curves of the type $y = g_k(x)$ which are tangent to a curve $C$ of the type $y^2 = g_n(x)$ at any fixed point $(\alpha, \beta)$ and at $k-1$ other points, is $C_k^k$ for $k > n/2$ and $\frac{1}{2}C_k^k$ for $k = n/2$, provided that the zeros of $g_n(x)$ are distinct.

For by Theorem 1 we get a curve of this type corresponding to any $k$ of the zeros of $g_n(x)$ and by Theorem 2 all curves of this type are obtained by this process. It must be shown, therefore, that when $k > n/2$ the same curve cannot be obtained from two different sets of $k$ zeros of $g_n(x)$. Suppose $y = \phi_k(x)$ and $y = \psi_k(x)$ are both of degree $k$ and cut out the same set of $k$ points $(\alpha_i, \beta_i)$ on $C$; then $\phi_k(x)$ and $\psi_k(x)$ must have at least one zero in common and therefore $\phi_k(x) = \psi_k(x)$. If $n$ is even and $k = \frac{1}{2}n$, then each set of $k$ such points is cut out by two and only two of these curves by Theorem 1.

From Theorem 1, the ordinary construction for drawing a
tangent to a conic at a point \( P \) on it, when the axes and vertices are known, follows immediately.

The following example is a rather interesting illustration of Theorem 1. Let \( C \) be the curve

\[
y^2 = f_\theta(x) = -x^6 + 14x^4 - 49x^2 + 36.
\]

The zeros of \( f_\theta(x) \) are 1, \(-1\), 2, \(-2\), 3, \(-3\). Let the curve \( c_1 : y = f_\theta(x) \) be determined by \((0, 6)\) \((1, 0)\) \((-1, 0)\) \((3, 0)\) and the curve \( c_2 : y = g_\theta(x) \) be determined by \((0, 6)\) \((2, 0)\) \((-2, 0)\) \((-3, 0)\); then we have

\[
\begin{align*}
f_\theta(x) &= 2x^3 - 6x^2 - 2x + 6, \\
g_\theta(x) &= -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x + 6.
\end{align*}
\]

These curves \( c_1 \) and \( c_2 \) meet on \( C \) in three points whose abscissas are 0, \((9 + \sqrt{241})/10\), \((9 - \sqrt{241})/10\). The curve \( S \) whose equation is

\[
y = \frac{f_\theta(x) + g_\theta(x)}{2} = \frac{3}{4}x^3 - \frac{15}{4}x^2 + 6
\]

is tangent to \( C \) at each of these three points.

If we take for \( c_1 \) the curve \( y = g_\theta(x) \) determined by \((0, 6)\), and for \( c_2 \) the curve \( y = g_\theta(x) \) determined by \((0, 6)\) \((1, 0)\) \((-1, 0)\) \((2, 0)\) \((-2, 0)\) \((3, 0)\) \((-3, 0)\), we get

\[
\begin{align*}
g_\theta(x) &= 6, \\
g_\theta(x) &= -\frac{1}{6}x^6 + \frac{7}{3}x^4 - \frac{49}{6}x^2 + 6.
\end{align*}
\]

The curves \( c_1 \) and \( c_2 \) are each tangent to \( C \) at each of the three points \((0, 6)\) \((\sqrt{7}, 6)\) \((-\sqrt{7}, 6)\) and the curve \( S \) whose equation is

\[
y = \frac{g_\theta(x) + g_\theta(x)}{2} = -\frac{1}{12}x^6 + \frac{7}{6}x^4 - \frac{49}{12}x^2 + 6
\]

meets \( C \) four times at each of the three points.