

APPLICATION OF BOOLEAN ALGEBRA TO PROVING
CONSISTENCY AND INDEPENDENCE
OF POSTULATES*

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1. *Object.* The usual procedure in proving the consistency of a set of postulates is to find such an interpretation of the undefined ideas involved in the postulates as to make all the postulates true propositions. The procedure in proving the independence of a particular postulate from the rest of the set is to find such an interpretation of the undefined ideas as to make that postulate, and only that one, false. But there exists no *method* of finding these interpretations. The object of this paper is to show how Boolean algebra partially provides such a method.

2. *General Procedure.* The postulates of a mathematical science are conditions on the classes, operations, and relations constituting the undefined ideas of the science. There is an infinitude of Boolean algebras.† In each of these algebras there is, for every positive integer n , the general n -ary operation

$$(I) \quad Axy \cdots t + Bxy \cdots t' + \cdots + Lx'y' \cdots t'‡$$

and the general n -adic relation

$$(II) \quad Axy \cdots t + Bxy \cdots t' + \cdots + Lx'y' \cdots t' = 0.$$

The procedure constituting a partial method of proving the consistency and the independence of a set of postulates consists, in the case of a consistency proof, in obtaining the conditions on the *discriminants* of (I) and (II) which will make (I) and (II) satisfy all the given postulates; if these Boolean conditions are consistent,§ the given postulates are consistent. In the case of

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† There is a Boolean algebra the cardinal number of whose elements is that of the continuum; one whose elements are denumerably infinite; and one for every positive integer n , the number of whose elements is 2^n .

‡ The usual Boolean notation is used, with a' denoting the negative of a .

§ The determination of the consistency of a set of Boolean relations is, as is well known, always possible. The vanishing of the product of the discriminants of the single relation equivalent to the set is the necessary and sufficient condition for this consistency.

proof of the independence of a postulate P from the other postulates, the procedure consists in obtaining the conditions on the discriminants of (I) and (II) which will make (I) and (II) contradict P and not contradict any of the other postulates; if these Boolean conditions are consistent, postulate P is independent of the other postulates. If in the case of a consistency proof the resulting Boolean conditions are inconsistent, the consistency of the postulates remains undetermined. Similarly for the case of an independence proof. It is this possibility of an inconsistent set of resulting Boolean conditions which makes my procedure only a partial method for determining consistency and independence.

It is to be observed that the method here outlined does not require the actual finding of the desired Boolean consistency and independence systems, though such systems, when the Boolean conditions are consistent, can always be found.* The consistency of the resulting Boolean conditions proves that a desired Boolean proof-system *can* be found, and this possibility is sufficient for our purpose.

It is to be observed also that the conditions on the discriminants of (I) and (II) corresponding to a given postulate may be used in the investigation of consistency and independence of any set of propositions involving that postulate. At the same time, these conditions give valuable information concerning the operations and relations of Boolean algebra.

The proposed procedure of investigating consistency and independence of postulates thus gives results that are generally of far wider application than the results obtained by the usual methodless procedure.

The method outlined above will be made clearer by applying it to a set of postulates for abelian groups and to a set of postulates for series.

* The desired proof-system is simply the system in which the elements are the elements of the Boolean algebra employed, and the operations and relations are the operations and relations of forms (I) and (II) *with the discriminants satisfying the conditions in question*. Thus, the Boolean consistency system for postulates $P_1 - P_3$ of §3 below, is the system (K, \circ) in which K is the class of elements of any Boolean algebra and $a \circ b$ is the operation (α) with the discriminants satisfying condition (4).

3. *Application to Postulates for Abelian Groups.* An *abelian group* is a class K containing a binary operation \circ satisfying postulates $P_1 - P_3$ following:*

P_1 . $a \circ (b \circ c) = (a \circ b) \circ c$, whenever the indicated elements belong to K .

P_2 . For any two elements a, b of K there is an element x of K such that $a \circ x = b$.

P_3 . $a \circ b = b \circ a$, whenever the indicated elements belong to K .

To investigate the consistency and the independence of $P_1 - P_3$, consider the general binary Boolean operation

$$(\alpha) \quad Aab + Bab' + Ca'b + Da'b'.$$

The necessary and sufficient conditions that (α) satisfy P_1, P_2, P_3 are, respectively:†

$$(1) \quad D = AD, (BC' + B'C)(AD + A'D') = 0,$$

$$(2) \quad B = A', C = D',$$

$$(3) \quad B = C.$$

From (1) - (3), it is seen that (α) will contradict P_1, P_2, P_3 if, respectively,

$$(1)' \quad DA' = 1,$$

$$(2)' \quad AB + A'B' = 1,$$

$$(3)' \quad BC' + B'C = 1.$$

Relations (1) - (3) are together equivalent to the relation

$$(4) \quad f(A, B, C, D) \equiv DA' + AB + A'B' + CD + C'D' + BC' + B'C = 0.$$

Relations (1)', (2), (3) are together equivalent to

$$(5) \quad f_1(A, B, C, D) \equiv 1 = 0.$$

Relations (1), (2)', (3) together are equivalent to

$$(6) \quad f_2(A, B, C, D) \equiv DA' + AB' + A'B + BC' + B'C = 0.$$

* See Transactions of this Society, vol. 4 (1903), p. 27.

† See Transactions of this Society, vol. 26 (1924), pp. 171-173.

Finally, relations (1), (2), (3)' together are equivalent to

$$(7) \quad f_3(A, B, C, D) \equiv A' + B + C' + D = 0.$$

Since the discriminant $f(0, 1, 1, 0)$ vanishes, condition (4) is possible. Hence, postulates $P_1 - P_3$ are consistent.

Condition (5) is impossible, and the question of the independence of postulate P_1 remains unsettled.*

Since $f_2(1, 1, 1, 1) = 0$, condition (6) is possible. Hence P_2 is independent of the other postulates.

Finally, since $f_3(1, 0, 1, 0) = 0$, condition (7) is possible. Hence, P_3 is independent of the other postulates.

4. *Application to Postulates for Series.*† The elements a, b, c, \dots of a class K form a *series* with respect to a dyadic relation S , if they satisfy the postulates $Q_1 - Q_3$ following:

Q_1 . If $a \neq b$, then either aSb or bSa .

Q_2 . If aSb , then $a \neq b$.

Q_3 . If aSb and bSc , then aSc .

Consider the general dyadic Boolean relation

$$(\beta) \quad Aab + Bab' + Ca'b + Da'b' = 0.$$

This relation, I have shown, can be a serial relation only if the Boolean algebra is a two-element one. Let (β) , then, be the general dyadic relation in a two-element Boolean algebra.

The necessary and sufficient conditions that (β) satisfy Q_1 and Q_2 are, respectively:

$$(i) \quad BC = 0,$$

$$(ii) \quad AD = 1.$$

The necessary and sufficient condition that (β) satisfy Q_3 non-vacuously is

$$(iii) \quad (A + D)B'C' = 0, \quad AD = 0.$$

The necessary and sufficient condition that (β) satisfy Q_3 vacuously is

$$(iv) \quad AD(B + C) = 1.$$

* That the denial of (1) is inconsistent with (2) and (3) may also be seen by observing that (2) and (3) imply (1).

† For the results through condition (iv) below, see this Bulletin, vol. 32 (1926), pp. 523-524.

From (i)–(iii), relation (β) will contradict Q_1, Q_2, Q_3 if, respectively:

- (i)' $BC = 1,$
 (ii)' $AD = 0,$
 (iii)' $AD = 1.$

Conditions (i)–(iii) are together equivalent to

- (v) $1 = 0.$

Conditions (i), (ii), (iv) are together equivalent to

- (vi) $F(A, B, C, D) \equiv BC + A' + D' + B'C' = 0.$

Conditions (i)', (ii), (iv) are together equivalent to

- (vii) $F_1(A, B, C, D) \equiv B' + C' + A' + D' = 0.$

Conditions (i), (ii)', (iii) are together equivalent to

- (viii) $F_2(A, B, C, D) \equiv BC + AD + (A + D)B'C' = 0.*$

Finally, conditions (i), (ii), (iii)' are together equivalent to

- (ix) $F_3(A, B, C, D) \equiv BC + A' + D' = 0.$

Relation (v) is impossible. Hence, in a two-element Boolean algebra there is no relation (β) which satisfies $Q_1 - Q_3$ non-vacuously, and the question of the consistency of $Q_1 - Q_3$ remains unsettled.

But relation (vi) is possible, since the discriminant $F(1, 1, 0, 1)$ vanishes. Hence, postulates $Q_1 - Q_3$ are consistent.

Relation (vii) is possible, since $F_1(1, 1, 1, 1) = 0$. Hence postulate Q_1 is independent of the other postulates.

Relation (viii) is possible, since $F_2(0, 0, 0, 0) = 0$. Hence Q_2 is independent of the other postulates.

Finally, relation (ix) is possible, since $F_3(1, 0, 0, 1) = 0$. Hence, Q_3 is independent of the other postulates.

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* If (iv) had been used here instead of (iii), the resulting condition would have been $1 = 0$, from which no conclusion could have been drawn with regard to the independence of Q_2 .