A NON-DENSE PLANE CONTINUUM†

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The author has shown ‡ that if M denotes a square plus its interior in a plane S, then there exists an upper semi-continuous collection § G of mutually exclusive non-degenerate subcontinua of M filling up M and such that G is homeomorphic with M. The present paper gives a continuum M *which contains no domain* yet which has the above property.

Let I denote the interior of a square J in a plane S. Let G be an upper semi-continuous collection of mutually exclusive non-degenerate continua filling $J+I$ such that G is homeomorphic with $J+I$. Since no element of G separates S it follows|| that if S' denotes the collection consisting of the elements of G and the *points* of S not belonging to any element of G, then S' corresponds to S under a continuous one to one correspondence U, and G corresponds to $J+I$. Let G^* denote the subcollection containing every element of G which contains a point of J. The set G^* is a simple closed curve. Moreover every element of G^* has in common with J either an arc or a single point. Then there exists¶ a continuous one to one correspondence U_1 between G^*

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‡ *On a problem of C. Kuratowski concerning upper semi-continuous collections*, Fundamenta Mathematicae, vol. 14 (1929), pp. 96–102.

§ For a definition of this term, and of the notion *limit element*, see R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

|| See R. L. Moore, loc. cit., Theorem 22.

¶ This may be shown as follows. Let $s_1, s_2, s_3, \cdots$ denote the maximal arcs which are subsets of J and which belong to some element of G^*. For each i and j $(i \neq j)$ the set $s_i \cdot s_j$ is vacuous. Let $v(s_i)$ denote the length of the interval s_i, and $v(J)$ the length of J. Suppose first that $v(J) - \sum_{i=1}^{\infty} v(s_i)$ is a positive number e and let d be a positive number less than e. A sequence of segments $t_1, t_2, t_3, \cdots$ can be defined inductively so that (1) for each i there is a j such that t_j contains s_i, (2) no two of the segments $t_1, t_2, t_3, \cdots$ have any point in common, and (3) $\sum_{i=1}^{\infty} v(t_i)$ is less than $v(J) - d$. Since the point set $J - \sum_{i=1}^{\infty} t_i$ has positive measure it is uncountable. Now the curve J can be transformed into itself in such a way that the sum of the lengths of the images of the intervals $s_1, s_2, s_3, \cdots$ is less than the length of J. Hence in any case there is a set of segments and J such that for uncountably many points x of J it is true that $U_1(x)$ (the element of G^* corresponding to the point x) contains x. The correspondence U_1 can† be extended so that there results a continuous one to one correspondence π between G and $J+I$ such that for uncountably many points P of J it is true that $\pi(P)$ contains P.

Now only a countable number of elements of G contain domains. Let H denote $J+I$ and let x denote any point of H. Let $C_1(x)$ denote x. Let $C_2(x)$ be the continuum $\pi(x)$. Let $C_3(x)$ denote the sum of all continua $\pi(y)$ for all points y of $C_2(x)$. In general let $C_{n+1}(x)$ denote the sum of all elements $\pi(y)$ for every point y of $C_n(x)$. If x and y are distinct points of H, then $C_n(x)$ and $C_n(y)$ are mutually exclusive continua. Hence, for each n, the set of all points x such that $C_n(x)$ contains a domain is countable. Hence there exist two points P_1 and P_2 of J, and a simple continuous arc T from P_1 to P_2, such that if x is any point of T then, for every n, $C_n(x)$ contains no domain, and $\pi(P_i)$ contains $P_i (i=1, 2)$.

There exists an infinite set of simple closed curves J_1, J_2, $J_3, \cdots$ such that (1) $J_1=J$, (2) $J_1 \cdot J_2$ is the point P_2, and in general, for each i, $J_i \cdot J_{i+1}$ is a single point P_{i+1}, and $J_i \cdot J_{i+k} = 0 (k>1)$, (3) no point of J_k lies within J_i, and (4) only a finite number of the curves J_1, J_2, $J_3, \cdots$ have points within any circle. For each i let π_i be a continuous transformation throwing

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$t_1, t_2, t_3, \cdots$ satisfying (1) and (2) above, and such that $J-\sum_{i=1}^{\infty} t_i$ is uncountable, and indeed if $v(J)-\sum_{i=1}^{\infty} v(s_i)$ is a positive number e, then the segments $t_1, t_2, t_3, \cdots$ can be so chosen that the measure of the set $J-\sum_{i=1}^{\infty} t_i$ is as near e as we please. If P is any point of $J-\sum_{i=1}^{\infty} t_i$, then let $C(P)$ denote P. Consider every interval s_j that is a subset of t_i as an element, and every other point of $\bar{t}_i$ as an element. Then the collection of elements so obtained is an arc, and can be made to correspond to the arc $\bar{t}_i$ of J. Thus if T denotes the collection of intervals $s_1, s_2, s_3, \cdots$ and all other points of J, then there exists a correspondence C such that (1) $C(T)=J$ and (2) for uncountably many points P of J the point P is an element of T, and $C(P)=P$. But if x is any element of T, then there is a continuum g_x of G^* containing x, and the correspondence D throwing g_x into x, for every element x of T, is continuous. Then if g is an element of G^* the correspondence throwing g into $C[D(g)]$ is a continuous one to one correspondence between G^* and J and satisfies the required conditions.

† See Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286–328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529–531.

H into J_i plus its interior in such a way that $\pi_i(P_1)=P_i$, and $\pi_i(P_2)=P_{i+1}$. For each n let E_n denote the sum of all continua $C_n(x)$ for every point x of the arc T. Note that E_{n+1} can be obtained by adding together all continua $\pi(y)$ for all points y of E_n. Let M_n denote $\pi_n(E_n)$ and let M denote $M_1+M_2+M_3+\cdots$. Then M is the continuum desired.

Since no one of the sets $M_1, M_2, M_3, \cdots$ contains a domain, the continuum M contains no domain. Let R denote $P_1+P_2+P_3+\cdots$ and let x denote any point of $M-R$ belonging to $M_i (i>1)$. Let y_x be the point of H such that $\pi_i(y_x)=x$. Let g_{y_x} denote the element of G corresponding, under π, to the point y_x, and let h_x denote the continuum $\pi_i(g_{y_x})$. In case h_x does not contain P_i or P_{i+1}, then let k_x denote h_x. For each i $(i>2)$ there is a point x_{i-1} of M_{i-1}, and a point $\bar{x}_i$ of M_i such that both the sets $h_{x_{i-1}}$ and h_{x_i} contain P_i. Let k_{P_i} denote $h_{x_{i-1}}+h_{\bar{x}_i}$. For some point x of M_2 the set h_x contains P_2. Let k_{P_2} denote h_x plus the arc T. Then M is the sum of the elements of an upper semi-continuous collection G' of mutually exclusive continua, every element of G' being a continuum k_Q for some point Q of M. The elements $k_{P_3}, k_{P_4}, k_{P_5}, \cdots$ are each homeomorphic with the sum of two elements of G and every other element of G' except k_{P_2} is homeomorphic with some one element of G. Every element of G' is a nondegenerate continuum. For each i $(i>1)$ let G_i' denote the collection of all elements k_P of G' for all points P of M_i. Then G_2' is homeomorphic with the arc M_1, G_3' is homeomorphic with M_2, G_4' with M_3, and so on indefinitely. Moreover, G_i' and G_{i+1}' have exactly one element in common, which corresponds to the common point of M_{i-1} and $M_i (i>1)$. Thus G and M are homeomorphic.

A bounded continuum with the same property may be obtained if condition (4) satisfied by the curves $J_1, J_2, J_3, \cdots$ is replaced by the following: (4) $J_1, J_2, J_3, \cdots$ is a contracting sequence having P_1 as sequential limit point. A continuum so obtained will have exactly two complementary domains in the plane S.

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