

A NON-DENSE PLANE CONTINUUM†

BY J. H. ROBERTS

The author has shown † that if M denotes a square plus its interior in a plane S , then there exists an upper semi-continuous collection § of mutually exclusive non-degenerate subcontinua of M filling up M and such that G is homeomorphic with M . The present paper gives a continuum M which contains no domain yet which has the above property.

Let I denote the interior of a square J in a plane S . Let G be an upper semi-continuous collection of mutually exclusive non-degenerate continua filling $J+I$ such that G is homeomorphic with $J+I$. Since no element of G separates S it follows || that if S' denotes the collection consisting of the elements of G and the points of S not belonging to any element of G , then S' corresponds to S under a continuous one to one correspondence U , and G corresponds to $J+I$. Let G^* denote the subcollection containing every element of G which contains a point of J . The set G^* is a simple closed curve. Moreover every element of G^* has in common with J either an arc or a single point. Then there exists ¶ a continuous one to one correspondence U_1 between G^*

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‡ On a problem of C. Kuratowski concerning upper semi-continuous collections, *Fundamenta Mathematicae*, vol. 14 (1929), pp. 96-102.

§ For a definition of this term, and of the notion limit element, see R. L. Moore, *Concerning upper semi-continuous collections of continua*, *Transactions of this Society*, vol. 27 (1925), pp. 416-428.

|| See R. L. Moore, loc. cit., Theorem 22.

¶ This may be shown as follows. Let s_1, s_2, s_3, \dots denote the maximal arcs which are subsets of J and which belong to some element of G^* . For each i and j ($i \neq j$) the set $s_i \cdot s_j$ is vacuous. Let $v(s_i)$ denote the length of the interval s_i , and $v(J)$ the length of J . Suppose first that $v(J) - \sum_{i=1}^{\infty} v(s_i)$ is a positive number e and let d be a positive number less than e . A sequence of segments t_1, t_2, t_3, \dots can be defined inductively so that (1) for each i there is a j such that t_j contains s_i , (2) no two of the segments t_1, t_2, t_3, \dots have any point in common, and (3) $\sum_{i=1}^{\infty} v(t_i)$ is less than $v(J) - d$. Since the point set $J - \sum_{i=1}^{\infty} t_i$ has positive measure it is uncountable. Now the curve J can be transformed into itself in such a way that the sum of the lengths of the images of the intervals s_1, s_2, s_3, \dots is less than the length of J . Hence in any case there is a set of segments

and J such that for uncountably many points x of J it is true that $U_1(x)$ (the element of G^* corresponding to the point x) contains x . The correspondence U_1 can† be extended so that there results a continuous one to one correspondence π between G and $J+I$ such that for uncountably many points P of J it is true that $\pi(P)$ contains P .

Now only a countable number of elements of G contain domains. Let H denote $J+I$ and let x denote any point of H . Let $C_1(x)$ denote x . Let $C_2(x)$ be the continuum $\pi(x)$. Let $C_3(x)$ denote the sum of all continua $\pi(y)$ for all points y of $C_2(x)$. In general let $C_{n+1}(x)$ denote the sum of all elements $\pi(y)$ for every point y of $C_n(x)$. If x and y are distinct points of H , then $C_n(x)$ and $C_n(y)$ are mutually exclusive continua. Hence, for each n , the set of all points x such that $C_n(x)$ contains a domain is countable. Hence there exist two points P_1 and P_2 of J , and a simple continuous arc T from P_1 to P_2 , such that if x is any point of T then, for every n , $C_n(x)$ contains no domain, and $\pi(P_i)$ contains P_i ($i = 1, 2$).

There exists an infinite set of simple closed curves J_1, J_2, J_3, \dots such that (1) $J_1 = J$, (2) $J_1 \cdot J_2$ is the point P_2 , and in general, for each i , $J_i \cdot J_{i+1}$ is a single point P_{i+1} , and $J_i \cdot J_{i+k} = 0$ ($k > 1$), (3) no point of J_k lies within J_i , and (4) only a finite number of the curves J_1, J_2, J_3, \dots have points within any circle. For each i let π_i be a continuous transformation throwing

t_1, t_2, t_3, \dots satisfying (1) and (2) above, and such that $J - \sum_{i=1}^{\infty} i t_i$ is uncountable, and indeed if $v(J) - \sum_{i=1}^{\infty} i v(s_i)$ is a positive number ϵ , then the segments t_1, t_2, t_3, \dots can be so chosen that the measure of the set $J - \sum_{i=1}^{\infty} i t_i$ is as near ϵ as we please. If P is any point of $J - \sum_{i=1}^{\infty} i t_i$, then let $C(P)$ denote P . Consider every interval s_j that is a subset of t_i as an element, and every other point of t_i as an element. Then the collection of elements so obtained is an arc, and can be made to correspond to the arc \bar{t}_i of J . Thus if T denotes the collection of intervals s_1, s_2, s_3, \dots and all other points of J , then there exists a correspondence C such that (1) $C(T) = J$ and (2) for uncountably many points P of J the point P is an element of T , and $C(P) = P$. But if x is any element of T , then there is a continuum g_x of G^* containing x , and the correspondence D throwing g_x into x , for every element x of T , is continuous. Then if g is an element of G^* the correspondence throwing g into $C[D(g)]$ is a continuous one to one correspondence between G^* and J and satisfies the required conditions.

† See Schoenflies, *Beiträge zur Theorie der Punktmengen*, *Mathematische Annalen*, vol. 62 (1906), pp. 286–328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, *Proceedings of the National Academy of Sciences*, vol. 6 (1920), pp. 529–531.

H into J_i plus its interior in such a way that $\pi_i(P_1) = P_i$, and $\pi_i(P_2) = P_{i+1}$. For each n let E_n denote the sum of all continua $C_n(x)$ for every point x of the arc T . Note that E_{n+1} can be obtained by adding together all continua $\pi(y)$ for all points y of E_n . Let M_n denote $\pi_n(E_n)$ and let M denote $M_1 + M_2 + M_3 + \dots$. Then M is the continuum desired.

Since no one of the sets M_1, M_2, M_3, \dots contains a domain, the continuum M contains no domain. Let R denote $P_1 + P_2 + P_3 + \dots$ and let x denote any point of $M - R$ belonging to $M_i (i > 1)$. Let y_x be the point of H such that $\pi_i(y_x) = x$. Let g_{y_x} denote the element of G corresponding, under π , to the point y_x , and let h_x denote the continuum $\pi_i(g_{y_x})$. In case h_x does not contain P_i or P_{i+1} , then let k_x denote h_x . For each $i (i > 2)$ there is a point x_{i-1} of M_{i-1} , and a point \bar{x}_i of M_i such that both the sets $h_{x_{i-1}}$ and $h_{\bar{x}_i}$ contain P_i . Let k_{P_i} denote $h_{x_{i-1}} + h_{\bar{x}_i}$. For some point x of M_2 the set h_x contains P_2 . Let k_{P_2} denote h_x plus the arc T . Then M is the sum of the elements of an upper semi-continuous collection G' of mutually exclusive continua, every element of G' being a continuum k_Q for some point Q of M . The elements $k_{P_1}, k_{P_2}, k_{P_3}, \dots$ are each homeomorphic with the sum of two elements of G and every other element of G' except k_{P_2} is homeomorphic with some one element of G . Every element of G' is a nondegenerate continuum. For each $i (i > 1)$ let G'_i denote the collection of all elements k_P of G' for all points P of M_i . Then G'_i is homeomorphic with the arc M_1 , G'_2 is homeomorphic with M_2 , G'_3 with M_3 , and so on indefinitely. Moreover, G'_i and G'_{i+1} have exactly one element in common, which corresponds to the common point of M_{i-1} and $M_i (i > 1)$. Thus G and M are homeomorphic.

A bounded continuum with the same property may be obtained if condition (4) satisfied by the curves J_1, J_2, J_3, \dots is replaced by the following: (4) J_1, J_2, J_3, \dots is a contracting sequence having P_1 as sequential limit point. A continuum so obtained will have exactly two complementary domains in the plane S .

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