A NOTE ON CYCLIC ALGEBRAS OF ORDER SIXTEEN

BY A. A. ALBERT*

1. *Introduction*. In a recent paper† I considered cyclic (Dickson) algebras of order sixteen generated by a cyclic quartic field Z and a quantity γ in the reference field F. It was proved there that if the algebra A were a division algebra and if γ^2 were the norm of a quantity of Z, so that the Wedderburn *norm condition* would not be satisfied, then A would be the direct product of two generalized quaternion algebras. It was not proved, however, that such division algebras existed.

R. Brauer has recently‡ proved that there exist normal division algebras which are direct products of two generalized quaternion algebras. In the present note I give an example of a *cyclic* algebra over the Brauer reference field for which the *norm condition* is not satisfied, therefore completing the theory of the previous paper.

2. *The Example*. Let $F=R(\xi, \eta)$, where ξ and η are indeterminates and R is the field of all rational numbers. This is the reference field of the algebras of Brauer. We shall use the notations of my paper (loc. cit.), Theorem 3. It was proved there that a necessary and sufficient condition that a direct product of two generalized quaternion algebras over F be a division algebra is that the connected form

$$\tau x_1^2 + \sigma x_2^2 - \sigma\tau x_3^2 - (\gamma x_4^2 + \rho x_5^2 - \rho\gamma x_6^2), \tag{1}$$

in the variables $x_1, \cdots, x_6$ in F be not a null form. We shall take

$$\sigma = -2\xi^3, \quad \rho = \eta, \quad \gamma = -1, \quad \tau = \alpha, \tag{2}$$

where α is a rational number not the square of a rational number.

Suppose that $\alpha = \nu^2$, where ν is in F. Then we may write

$$\nu = bc^{-1},$$

* Presented to the Society, September 9, 1931.

† This Bulletin, vol. 37 (1931), pp. 301–312.

‡ Mathematische Zeitschrift, vol. 31 (1930), §5. Brauer's example is that of an algebra not necessarily a cyclic algebra and it would probably be difficult to prove it cyclic even if this were the case. Also his proof that the algebra is a division algebra is essentially different from ours.

where $b=b(\xi, \eta)$ and $c=c(\xi, \eta)$ are polynomials in ξ and η with coefficients in R such that the greatest common divisors of b and c are rational numbers. Then

$$b^2 = \alpha c^2, \quad b_1^2 = \alpha c_1^2,$$

where $b_1=b(\xi, 0)$, $c_1=c(\xi, 0)$. The coefficient of the highest power of ξ in b_1^2 is evidently a rational square, that in $c_1^2\alpha$ not such a square, since α is not a rational square. Hence $b_1^2=c_1^2\alpha$ implies that $b_1=c_1=0$ and b and c have the common factor η, a contradiction.

Suppose that (1) vanished for $x_1, \cdots, x_6$ not all zero and in F. Evidently we could take $x_1, \cdots, x_6$ to be polynomials in ξ and η with rational coefficients and with greatest common divisor a rational number. We thus write

$$x_i = x_i(\xi, \eta),$$

with rational coefficients and having no factor in common. Equation (1) becomes

$$\alpha x_1^2 - 2\xi^3 x_2^2 + 2\alpha\xi^3 x_3^2 = - x_4^2 + \eta(x_5^2 + x_6^2), \tag{3}$$

which must be satisfied identically in ξ and η so that it must be satisfied when we replace η by zero. Let $x_i(\xi, 0)=y_i$ and let the highest power of ξ occurring in y_i be ξ^{r_i}, its coefficient being λ_i where $\lambda_i=0$ if and only if $y_i=0$. Then (3) becomes

$$y_4^2 + \alpha y_1^2 = 2\xi^3(y_2^2 - \alpha y_3^2), \tag{4}$$

identically in ξ. According as $r_4>r_1$, $r_4<r_1$, $r_4=r_1$, the term of highest degree in $y_4^2+\alpha y_1^2$ is

$$\lambda_4^2\xi^{2r_4}, \quad \alpha\lambda_1^2\xi^{2r_1}, \quad (\lambda_4^2 + \alpha\lambda_1^2)\xi^{2r_1},$$

an even power, and is zero if and only if $y_1=y_4=0$. Similarly the possible terms of highest degree in $2\xi^3(y_2^2-\alpha y_3^2)$ are

$$2\lambda_2^2\xi^{2r_2+3}, \quad -2\alpha\lambda_3^2\xi^{2r_3+3}, \quad 2(\lambda_2^2 - \alpha\lambda_3^2)\xi^{2r_2+3},$$

so that (3) and its consequence (4) imply that a polynomial whose degree is even is equal to a polynomial whose degree is odd. This is possible only when both polynomials are zero, so that $\lambda_4=\lambda_1=\lambda_2=\lambda_3=0$ and hence $y_1=y_2=y_3=y_4=0$. But then $x_1, \cdots, x_4$ are divisible by η, so that we may write

$$x_i = \eta z_i, \qquad (i = 1, \cdots, 4)$$

and obtain

$$\eta^2(z_1^2 - 2\xi^3 z_2^2 + 2\alpha z_3^2 + z_4^2) = \eta(x_5^2 + x_6^2), \tag{5}$$

from (3). It follows that $x_4^2+x_6^2$ is divisible by η. Let the constant term with respect to η of x_5 be μ_5, that of x_6 be μ_6. Then obviously $\mu_5^2+\mu_6^2=0$, where μ_5 and μ_6 are polynomials in ξ with rational coefficients. If $\mu_5\neq 0$, then $-1=(\mu_6\mu_5^{-1})^2$, which is impossible as we have shown, since -1 is not a rational square. Hence $\mu_5=\mu_6=0$ and each of the quantities x_5, x_6 is divisible by η. But then $x_1, \cdots, x_6$ are all divisible by η, a contradiction of our hypothesis. Hence we have proved that (3) is not a null form, *the direct product of our two generalized quaternion algebras*

$$B = (1, u, s, us), \quad us = -su, \quad u^2 = \tau, \quad s^2 = \sigma,$$
$$C = (1, j, t, jt), \quad tj = -jt, \quad j^2 = \gamma, \quad t^2 = \rho,$$

is a division algebra.

We now take $\alpha=\tau=1+\Delta^2$, where Δ is a rational number so chosen that $1+\Delta^2$ is not a rational square. For example Δ may be taken to be unity. Let

$$\beta_1 = \xi, \quad \gamma_0 = -\beta_1^2, \quad \nu = -\eta(2\alpha\xi^2)^{-1}.$$

The author has shown that the equation

$$\phi(\omega) \equiv \omega^4 + 2\nu(1+\Delta^2)\omega^2 + \nu^2\Delta^2(1+\Delta^2) = 0 \tag{6}$$

is a cyclic quartic over F for every $\nu\neq 0$ in F and $\tau=1+\Delta^2$ not the square of any quantity of F. Our above choices of ν and τ evidently insure that these requirements are satisfied. I have also shown (loc. cit.) that if we define the cyclic algebra with the basis

$$x^r y^s, \qquad (r, s = 0, 1, 2, 3),$$

such that

$$\phi(x) = 0, \quad y^r x = \theta^r(x) y^r, \quad y^4 = \gamma_0, \qquad (r = 0, 1, \cdots),$$

then, if $\gamma_0=-\beta_1^2$, by Theorem 4 the algebra is expressible as the direct product of two generalized quaternion algebras B and C with

$$\tau = 1 + \Delta^2,\ \sigma = 2\beta_1\gamma_0,\ \rho = 2\nu\beta_1\tau(-\beta_1),\ \gamma = \gamma_0 = -\beta_1^2.$$

But $2\beta_1\gamma_0 = -2\xi^3$, $\rho = \eta$ and we may evidently choose a new basis with $\gamma = \gamma_0$ replaced by -1. Hence $B \times C$ is a division algebra. Moreover $\gamma_0 = -\beta_1^2$, $\gamma_0^2 = \beta_1^4$ is the norm of the scalar β_1 in $F(x)$. *For this cyclic algebra the Wedderburn norm condition is not satisfied and yet the algebra is a division algebra.*

THE UNIVERSITY OF CHICAGO

AN ACKNOWLEDGMENT

BY G. W. STARCHER

Since the publication of my article entitled *A note on geometrical factorial series* in the June issue of this Bulletin,* my attention has been called to the fact that most of the results of that paper are not new and have been previously given by F. Ryde.† Ryde considers a series which is essentially the same as (1) and calls it a "geometric factorial series." The term *geometric factorial series* was suggested to me by the fact that Cauchy had called the denominators of the terms in the series (1) *geometric factorials.*‡ Later *geometric* was changed to *geometrical.*

The conclusions in §2 are given by Ryde (page 6) and the comparison of (1) and (2) was evidently known to him. The theorem in §3 is proved in essentially the same way by Ryde (pages 6–8), and on page 8 a more general expansion is given. On pages 11–13 he employs the multiplication of such series, and he probably had formulas for the coefficients in such a product.

OHIO UNIVERSITY

---

* This Bulletin, vol. 37 (1931), pp. 455–463.

† *A contribution to the theory of linear homogeneous geometric difference equations* (*q-difference equations*), Lund dissertation, published by Lindstedts Univ. Bokhandel (1921).

‡ Encyclopédie des Sciences Mathématiques, vol. I, 4, p. 279.