ON A COVARIANT DIFFERENTIATION PROCESS*

BY H. V. CRAIG

1. *Introduction.* If the components of a tensor T^α_β are point functions, a tensor of one higher covariant order may be formed by differentiation and elimination. Specifically, the equations which express the law of transformation of the tensor are differentiated once with respect to each of the coordinate variables of a given set, and certain second derivatives appearing are eliminated by means of a relationship known as the fundamental affine connection. This process is called covariant differentiation and is one of the cardinal operations of the tensor calculus.†

It is the purpose of this note to point out that a somewhat similar process exist sfor tensors whose components are functions of x, dx/dt, d^2x/dt^2.

2. *Notation.* We shall suppose that the curves involved in the following discussions are given in parametric form and shall employ primes to indicate differentiation with respect to the parameter. Partial derivatives, for the most part, will be denoted by means of subscripts. Thus, we shall write x', x'', $\left\{ {\alpha \atop \beta} \right\}_{x'^\gamma}$ for dx/dt, d^2x/dt^2, $\partial \left\{ {\alpha \atop \beta} \right\} / \partial x'^\gamma$ respectively.

3. *The Differentiation Process.* We shall illustrate this process by applying it to a mixed tensor of the second order. Accordingly, suppose that the quantities $T^\alpha_\beta(x, x', x'')$ are defined along a given regular curve and are the components of a tensor of the type indicated. The extended point transformation,

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(1) \qquad \begin{aligned} x^\alpha &= x^\alpha(y^1, y^2, \cdots, y^n), \qquad x'^\alpha = \frac{\partial x^\alpha}{\partial y^i} y'^i, \\ x''^\alpha &= \frac{\partial x^\alpha}{\partial y^i} y''^i + \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^i y'^j, \end{aligned}
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* Presented to the Society, September 11, 1931.

† For an exposition of tensor analysis and covariant differentiation in particular reference may be made to Oswald Veblen, *The Invariants of Quadratic Differential Forms*, 1927, Chapters 2, 3; L. P. Eisenhart, *Riemannian Geometry*, 1926, Chapter 1.

induces the following transformation in T_β^α,

$$(2) \qquad \bar{T}_j^i(y, y', y'') = T_\beta^\alpha(x, x', x'') \cdot \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^j} \cdot$$

Differentiating (2) with respect to y'^k and making use of (1), we obtain

$$(3) \quad \bar{T}^i_{jy'^k} = \left\{ T^\alpha_{\beta x'^\gamma} \cdot \frac{\partial x^\gamma}{\partial y^k} + T^\alpha_{\beta x''^\gamma} \cdot 2 \frac{\partial^2 x^\gamma}{\partial y^a \partial y^k} \cdot y'^a \right\} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^j} \cdot$$

From this equation and the relationship

$$(4) \qquad \overline{\left\{ \begin{matrix} l \\ k \end{matrix} \right\}} \frac{\partial x^\gamma}{\partial y^l} = \frac{\partial^2 x^\gamma}{\partial y^k \partial y^a} \cdot y'^a + \left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} \frac{\partial x^\delta}{\partial y^k},$$

$$\left(\left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} \equiv x'^\beta \Gamma^\gamma_{\delta\beta} + \tfrac{1}{2} x''^\sigma f_{\delta\beta\sigma} f^{\beta\gamma} \right),$$

which is due to J. H. Taylor,* we can eliminate the second derivatives. To facilitate this elimination we multiply (4) by $2T^\alpha_{\beta x''^\gamma} \cdot (\partial y^i/\partial x^\alpha)(\partial x^\beta/\partial y^j)$ and sum, noting that by virtue of (1) $T^\alpha_{\beta x''^\gamma}$ is a tensor. The result is

$$2\bar{T}^i_{jy''^l} \cdot \overline{\left\{ \begin{matrix} l \\ k \end{matrix} \right\}} = \left[T^\alpha_{\beta x''^\gamma} \cdot 2 \frac{\partial^2 x^\gamma}{\partial y^a \partial y^k} y'^a \right.$$
$$\left. + 2T^\alpha_{\beta x''^\gamma} \left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} \frac{\partial x^\delta}{\partial y^k} \right] \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^j} \cdot$$

By subtracting this last equation from (3) we obtain our conclusion:

$$\bar{T}^i_{jy'^k} - 2\bar{T}^i_{jy''^l} \cdot \overline{\left\{ \begin{matrix} l \\ k \end{matrix} \right\}}$$

is a tensor whose covariant order is one greater than that of the original, $\bar{T}_j^i$.

Some of the obvious properties of this process are: (a), if the components of the tensor do not contain y'' the process reduces

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* See J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27, p. 255, equation (21). For the definition and properties of $f^{\alpha\beta}$, $f_{\alpha\beta\gamma}$, $\Gamma^\gamma_{\alpha\beta}$ reference may be made to pp. 247, 248, 253, ibid.

to partial differentiation; (b), the rules of ordinary calculus for the differentiation of sums and products are conserved.

Unlike ordinary covariant differentiation, however, this process is commutative with itself. To assure ourselves on this point it will suffice, perhaps, to illustrate with an arbitrary scalar $S(x, x', x'')$. On performing the evident cancellations the difference between the covariant derivatives $S_{,\beta,\gamma}$ and $S_{,\gamma,\beta}$ reduces to

$$2S_{x''^\lambda}\left[\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x'^\beta} - \left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x'^\gamma} + 2\left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x''^\mu}\left\{\begin{matrix}\mu\\ \gamma\end{matrix}\right\} - 2\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x''^\mu}\left\{\begin{matrix}\mu\\ \beta\end{matrix}\right\}\right].$$

By expanding and making use of the relation $\Gamma^\lambda_{\gamma\alpha} = f^{\lambda\sigma}[\gamma\alpha, \sigma]$, we obtain

$$\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x'^\beta} = \Gamma^\lambda_{\gamma\beta} + x'^\alpha(f^{\lambda\delta}[\gamma\alpha, \delta])_{x'^\beta} + \tfrac{1}{2}x''^\alpha(f_{\alpha\delta\gamma}f^{\delta\lambda})_{x'^\beta},$$

$$-\left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x'^\gamma} = -\Gamma^\lambda_{\gamma\beta} - x'^\alpha(f^{\lambda\delta}[\beta\alpha, \delta])_{x'^\gamma} - \tfrac{1}{2}x''^\alpha(f_{\alpha\delta\beta}f^{\delta\lambda})_{x'^\gamma},$$

$$2\left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x''^\mu}\left\{\begin{matrix}\mu\\ \gamma\end{matrix}\right\} = f_{\beta\delta\mu}f^{\delta\lambda}[x'^\alpha f^{\mu\sigma}[\gamma\alpha, \sigma] + \tfrac{1}{2}x''^\alpha f_{\gamma\rho\alpha}f^{\rho\mu}],$$

$$-2\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x''^\mu}\left\{\begin{matrix}\mu\\ \beta\end{matrix}\right\} = -f_{\gamma\delta\mu}f^{\delta\lambda}x'^\alpha f[^{\mu\sigma}[\beta\alpha, \sigma] + \tfrac{1}{2}x''^\alpha f_{\beta\rho\alpha}f^{\rho\mu}].$$

By virtue of the relationships* $f_{\beta\delta\mu} = f_{\beta\delta x'^\mu}$, $f_{\beta\delta}f^{\delta\lambda} = \delta^\gamma_\beta$ we may write $f_{\beta\delta\mu}f^{\delta\lambda} = -f_{\delta\mu}f^{\delta\lambda}_{x'^\beta}$, and thus an alternative form for the right member of the third equation of the set above is

$$-x'^\alpha f^{\delta\lambda}_{x'^\beta}[\gamma\alpha, \delta] - \tfrac{1}{2}x''^\alpha f_{\gamma\delta\alpha}f^{\delta\lambda}_{x'^\beta}.$$

Finally, it follows from the definition† of $[\gamma\alpha, \delta]$ and the

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* See J. H. Taylor, loc. cit., p. 248, line 28, and p. 249, equation (7).

† J. H. Taylor, loc. cit., equation (6), p. 248.

identity* $x'^{\alpha}f_{\alpha\beta\gamma}=0$ that $[\gamma\alpha,\ \delta]_{x'\beta}-[\beta\alpha,\ \delta]_{x'\gamma}=0$ and hence the difference in question vanishes.

A moment's consideration of the m times extended point transformation will enable us to extend the process to tensors whose components involve derivatives of any order. Thus, if the components $T^{\alpha\cdots}_{\beta\cdots}$ of a given tensor are functions of $x, x', x'', \cdots, x^{(m)}$, then the quantities

$$T^{\alpha\cdots}_{\beta\cdots x^{(m-1)}\gamma} - m\cdot T^{\alpha\cdots}_{\beta\cdots x^{(m)}\lambda}\cdot \left\{ \begin{matrix} \lambda \\ \cdot \\ \gamma \end{matrix} \right\}$$

are the components of a tensor. The commutative property established above holds in this case also.

The University of Texas

ON THE DIVISIBILITY OF LOCALLY CONNECTED SPACES†

BY G. T. WHYBURN

1. *Introduction.* In this paper a property of connected, locally connected, separable metric spaces will be established which is stronger than that of *divisibility* in the sense of W. A. Wilson.‡ In order to distinguish our property from that of Wilson we shall use the term *strong divisibility.*

A space M will be said to be *strongly divisible* if for every pair of mutually exclusive closed and connected subsets A and B of M there exists a decomposition of M into three mutually exclusive sets R, F, and G, where R and G are connected and open and contain A and B, respectively, and where F is the common boundary of R and G, that is, $F=F(R)=F(G)$.

2. Theorem. *Every connected, locally connected, separable, metric space M is strongly divisible.*

* J. H. Taylor, loc. cit., equation (14), p. 253.

† Presented to the Society, December 31, 1930.

‡ According to Wilson, a space M is *divisible* if for every pair of mutually exclusive subcontinua A and B of M there exists a decomposition of M into two continua P and Q such that $P\cdot B=Q\cdot A=0$. See this Bulletin, vol. 36 (1930), p. 85. Wilson's theorem that every connected, locally connected, separable metric space is divisible is obviously an immediate corollary to our theorem below in §2.