QUADRATIC ADDITION THEOREMS FOR EVEN FUNCTIONS*

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The purpose of this paper is to show that if an even function be expansible in power series in some region about the origin, and satisfy a quadratic addition† theorem, the function is a linear fractional transformation of the Weierstrass "P" function or one of its degenerate forms.

P. D. Edwards‡ has shown that addition theorems are symmetric in $U=F(x)$ and $V=F(y)$. He has also shown the most general form of an addition theorem linear in $W=F(x+y)$. In this paper, equations quadratic in W are treated by means of the following theorem.

THEOREM 1. *An addition theorem for an even function is symmetric in the three variables U, V, W.*

For one may set

$$\begin{aligned} x+y&=s, & W&=F(x+y)=F(s) & &=U',\\ x&=s+t, & U&=F(x)=F(s+t) & &=W',\\ y&=-t, & V&=F(y)=F(-t)=F(t) & &=V'. \end{aligned}$$

Thus the equation, which is known to be symmetric in U, V, is shown to be symmetric in U, W so that the theorem follows.

In the same manner one easily proves the following companion theorem for odd functions.

THEOREM 2. *An addition theorem for an odd function is symmetric in the three variables U, V, $-W$.*

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† An *algebraic addition theorem* is a polynomial relation between $F(x)$, $F(y)$, and $F(x+y)$, with constant coefficients. For this paper the polynomial is irreducible. Weierstrass made addition theorems the basis for his study of elliptic functions. Osgood, *Allgemeine Funktionentheorie*, 1912, vol. 1, pp. 580–595; Forsyth, *Theory of Functions*, 1918, Chapter XIII; Hancock, *Theory of Elliptic Functions*, 1910.

‡ Abstract No. 8, this Bulletin, vol. 35 (1929), p. 453. The paper is available at the University of Indiana Library.

This theorem is illustrated by tan x and ctn x, but the form is lost for sin x and Jacobi's sn x, whose addition theorems are symmetric in U^2, V^2, W^2.

The equation whose solution we wish to study is the following, symmetric in the three variables U, V, W:

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\begin{aligned}
(1)\quad & a_0 + a_1(U + V + W) + a_2(UV + UW + VW) \\
& + a_3(U^2 + V^2 + W^2) + a_4 UVW \\
& + a_5(UV^2 + U^2V + UW^2 + U^2W + VW^2 + V^2W) \\
& + a_6(U^2V^2 + U^2W^2 + V^2W^2) + a_7 UVW(U + V + W) \\
& + a_8 UVW(UV + UW + VW) + a_9 U^2V^2W^2 = 0.
\end{aligned}
$$

This may also be written in powers of W as

$$(2)\qquad R_1(U, V)W^2 - 2R_2(U, V)W + R_3(U, V) = 0,$$

where R_1, R_2, R_3 are of degree not more than two in U or V.

Since we are dealing with even functions, changing y to $-y$ alters only W. Hence the two roots of this quadratic are $W = F(x+y)$, $F(x-y)$, and by the ordinary theorems regarding symmetric functions of the roots one has

$$(3)\, F(x + y) + F(x - y) = 2R_2/R_1, F(x + y)\cdot F(x - y) = R_3/R_1.$$

Thus two other interesting types of functional equations are seen to arise naturally and will be solved when R_1, R_2, R_3 are quadratics symmetric in U, V.

It has been shown* that, if the first term of the power series expansion of $F(x)$ is not a constant, there is in the functional equation a group of terms which form an equation having the first term of the series as a solution. Since first terms of the form $c_n x^n$, $n < -2$, or $n > 2$, would require the vanishing of all the coefficients a in (1), we have to consider only three possible first terms, $c_{-2}x^{-2}$, c_0, c_2x^2. In case the first term is c_0, one writes

$$U = U_1 + c_0$$

and the first term for U_1 will be c_2x^2. In case the first term is c_2x^2, one writes

$$U = 1/U'$$

* The author in American Mathematical Monthly, vol. 37, p. 70.

and the first term for U' will be $c_{-2}x^{-2}$. The transformed equation in U_1, V_1, W_1, or U', V', W' will still be of type (1). Hence we see that any solution of an equation (1) is a linear fractional transformation of a solution of a type (1) equation, whose series expansion begins with $c_{-2}x^{-2}$, $c_{-2}\neq 0$. For this reduced case, equation (1) contains the terms of the addition theorem for x^{-2} as its terms of highest degree and they must enter with a non-zero multiplier (chosen as unity). This gives our equation (1) as

$$\begin{aligned} a_0 &+ a_1(U+V+W) + a_2(UV+UW+VW) + a_3(U^2+V^2+W^2) \\ &+ a_4UVW + a_5(UV^2+U^2V+UW^2+U^2W+VW^2+V^2W) \\ &+ U^2V^2+U^2W^2+V^2W^2 - 2UVW(U+V+W) = 0. \end{aligned}$$

The solution has the expansion

$$U = F(x) = \frac{1}{x^2} + c_0 + c_2x^2 + c_4x^4 + \cdots .$$

For $x=y$, the roots of (2) become

$$F(2x), \qquad F(0)=\infty, \qquad \text{and} \qquad R_1(U,U)\equiv 0,$$

so that $a_5=a_3=0$. Equation (1) is further reduced to

$$\begin{aligned} (4)\quad a_0 &+ a_1(U+V+W) + a_2(UV+UW+VW) + a_4UVW \\ &+ U^2V^2+U^2W^2+V^2W^2 - 2UVW(U+V+W) = 0. \end{aligned}$$

Equation (4) is invariant in form under the transformation $U=U'+t$ and the cubic term becomes $(a_4-12t)U'V'W'$. When $t=a_4/12$, equation (4) becomes, if we omit accents,

$$\begin{aligned} (5)\quad b_0 &+ b_1(U+V+W) + b_2(UV+VW+UW) \\ &+ U^2V^2+U^2W^2+V^2W^2 - 2UVW(U+V+W) = 0. \end{aligned}$$

If we set $x=y$ in (5), substitute the power series for the solution, and after collecting terms equate to zero the coefficients of each power of x, the first four conditions on c_i are:

$$(6)\qquad \begin{aligned} &x^{-6}: && c_0 = 0,\\ &x^{-4}: && 10c_2 - b_2 = 0,\\ &x^{-2}: && 28c_4 - b_1 = 0,\\ &x^{0}: && 2b_0 + 21b_2c_2 - 90c_2^2 - 510c_6 = 0. \end{aligned}$$

Again if we set $y=2x$ in (5) and make the same computation, there results:

$$
(7)\qquad
\begin{aligned}
x^{-6}&: & c_0 &= 0,\\
x^{-4}&: & 10c_2 - b_2 &= 0,\\
x^{-2}&: & 28c_4 - b_1 &= 0,\\
x^{0}&: & 18b_0 + 289b_2c_2 - 1210c_2^2 - 6390c_6 &= 0.
\end{aligned}
$$

The conditions (6) and (7) require for consistency that $4b_0=b_2^2$. To complete the proof one notices that if $b_2=g_2/2$, $b_1=g_3$, $b_0=b_2^2/4=g_2^2/16$, the equation (1) has been reduced by means of linear fractional transformations to the well known addition theorem* for $P(x; g_2, g_3)$. In this connection the degenerate forms of $P(x; g_2, g_3)$ should be taken as $(2/3+\text{ctn}^2 x)$ and x^{-2}, as these have $c_0=0$.

It will be seen from a study of the addition theorem for $P(x; g_2, g_3)$ that every second-degree addition theorem having an even solution may be written

$$W = \frac{R_2(UV) \pm kU'V'}{R_1(UV)},$$

where U' and V' are the derivatives of U and V with respect to x and y respectively.

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* Since the addition theorems give rise to the differential equation by a direct computation, (see Forsyth, p. 357), it is one of the many ways of defining the P function.