TRANSFORMATIONS OF DOUBLE SEQUENCES WITH APPLICATION TO CESÀRO SUMMABILITY OF DOUBLE SERIES*

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1. *Introduction*. Of the definitions of convergence commonly employed for double series, only that due to Pringsheim† permits a series to converge conditionally. Therefore, in spite of any disadvantages which it may possess, this definition is better adapted than others to the study of many problems in double sequences and series.

Chief among the reasons why the theory of double sequences, under the Pringsheim definition of convergence, presents difficulties not encountered in the theory of simple sequences is the fact that a double sequence $\{x_{ij}\}$ may converge without x_{ij} being a bounded function of i and j. Thus it is not surprising that many authors in dealing with the convergence of double sequences should have restricted themselves to the class of bounded sequences, or in dealing with the summability of double series, to the class of series for which the function whose limit is the sum of the series is a bounded function of i and j. Without such a restriction, peculiar things may sometimes happen; for example, a double power series may converge with partial sum S_{ij} unbounded at a place exterior to its associated circles of convergence.

Nevertheless there are problems in the theory of double sequences and series where this restriction of boundedness as it has been applied is considerably more stringent than need be. It is the purpose of the present paper to prove a general theorem concerning the regularity of transformations of double sequences when the original double sequence is not necessarily bounded, and to apply this theorem to the question of consistency of Cesàro summability of double series. The theorem immediately suggests numerous extensions and generalizations: extensions of results already known for certain classes of double sequences

* Presented to the Society, September 8, 1931.

† See Pringsheim, Encyklopädie der mathematischen Wissenschaften, vol. I, A3, pp. 97–98.

and series to new and broader classes; generalizations of well known results concerning simple sequences to certain classes of double sequences. Some of these problems we hope to consider in a later paper.

In recent years a considerable number of writers have given attention to the question of transformations of multiple sequences or series and related topics; these include especially Bromwich and Hardy,* C. N. Moore,† Smail,‡ Kojima,§ Eversull,|| Robison,¶ Merriman,** and Mears.†† So far as we are aware, the only ones to obtain results of regularity or consistency without imposing the condition of boundedness on the original sequence or series are Kojima and Miss Mears.

It should be emphasized that one may at once extend the following work, without meeting additional difficulties, to multiple sequences or series of any order of multiplicity.

2. *Transformations of Double Sequences.* Let $\{x_{mn}\}$ be a double sequence and $\|a_{mnkl}\|$, $(m, n, k, l=1, 2, 3, \cdots)$, a four-dimensional matrix of numbers, real or complex, with $a_{mnkl}=0$, for $k>m$ or $l>n$ or both. Then the transformation

$$y_{mn} = \sum_{k=1,l=1}^{m,n} a_{mnkl}x_{kl},$$

which we term *a double A-transformation*, defines a new double

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* Bromwich and Hardy, *Some extensions to multiple series of Abel's theorem on the continuity of power series*, Proceedings of the London Mathematical Society, (2), vol. 2 (1904), pp. 161–189.

† C. N. Moore, *On convergence factors in double series and the double Fourier's series*, Transactions of this Society, vol. 14 (1913), pp. 73–104; *On convergence factors in multiple series*, ibid., vol. 29 (1927), pp. 227–238.

‡ Smail, *Summability of double series*, Annals of Mathematics, (2), vol. 21 (1920), pp. 221–223.

§ Kojima, *On the theory of double sequences*, Tôhoku Mathematical Journal, vol. 21 (1922), pp. 3–14.

|| Eversull, *On convergence factors in triple series and the triple Fourier's series*, Annals of Mathematics, (2), vol. 24 (1922–23), pp. 141–166.

¶ Robison, *Divergent double sequences and series*, Transactions of this Society, vol. 28 (1926), pp. 50–73.

** Merriman, *Concerning the summability of double series of a certain type*, Annals of Mathematics, (2), vol. 28 (1927), pp. 515–533; *A set of necessary and sufficient conditions for the Cesàro summability of double series*, ibid., (2), vol. 29 (1928), pp. 343–354.

†† Mears, *Riesz summability for double series*, Transactions of this Society, vol. 30 (1928), pp. 686–709.

sequence $\{y_{mn}\}$. *If the transformation has the property that it carries over every convergent sequence* $\{x_{mn}\}$ ***of a certain class*** *into a sequence* $\{y_{mn}\}$ *convergent to the same limit, it is said to be* ***a regular transformation for this class of sequences.*** We are interested in determining sufficient conditions for regularity of a double A-transformation for sequences of a certain class.*

It may be recalled that a simple A-transformation, or A-transformation for simple sequences, is defined by means of a matrix

$$\|a_{mk}\| = \left\| \begin{matrix} a_{11} & 0 & 0 & 0 \cdots \\ a_{21} & a_{22} & 0 & 0 \cdots \\ a_{31} & a_{32} & a_{33} & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot \cdots \end{matrix} \right\|,$$

and that the Silverman-Toeplitz necessary and sufficient conditions for regularity of this transformation are

$$(1) \qquad \begin{gathered} \lim_{m\to\infty} a_{mk} = 0, \qquad (k = 1, 2, 3, \cdots), \\ \lim_{m\to\infty} \sum_{k=1}^{m} a_{mk} = 1, \qquad \sum_{k=1}^{m} | a_{mk} | < K, \qquad (m = 1, 2, 3, \cdots). \end{gathered}$$

A double A-transformation will be said to be the "product" of two simple A-transformations, A' and A'', whenever we have $a_{mnkl} = a'_{mk}a''_{nl}$, $(m, n, k, l = 1, 2, 3, \cdots)$; we shall then write $A = A' \odot A''$. On account of the trend of development taken by the theory of summability of double series, we are particularly interested in double A-transformations of this character. Concerning them we now establish the following theorem.

THEOREM 1. *Let A' and A'' be any two regular simple A-transformations. Then the double A-transformation $A' \odot A''$ is regular for the class of double sequences of which each row is transformed by a'', and each column by A', into a bounded sequence.*

PROOF. It follows at once from the definition that every double A-transformation is distributive. Moreover, since A' and A'' are regular and the second of conditions (1) is consequently satisfied by each, it is seen that any constant sequence, $\{x_{mn}\} = \{c\}$,

* Conditions both necessary and sufficient for regularity of a double A-transformation for the class of *bounded* sequences have been found by Robison, loc. cit., p. 53.

is transformed by the double A-transformation $A'\odot A''$ into a sequence converging to c. Hence we may, without loss of generality, confine ourselves to proving that if $\{x_{mn}\}$ converges to *zero*, $\{y_{mn}\}$ also converges to zero; that is, that given any $\epsilon>0$, there exist positive numbers M, N such that

$$(2) \qquad |y_{mn}| = \left| \sum_{k=1,l=1}^{m,n} a_{mnkl}x_{kl} \right| < \epsilon,$$

for $m>M$, $n>N$. Let any $\epsilon>0$ be assigned. A' and A'' being regular, a'_{mk} and a''_{nl} satisfy the conditions (1), where one K may serve for both transformations. And since $\{x_{mn}\}$ converges to zero, there exist positive integers p, q such that we have

$$(3) \qquad |x_{mn}| < \epsilon/(4K^2),$$

for $m>p$, $n>q$. Let p, q be held fast; henceforth we restrict ourselves to values of $m>p$ and of $n>q$. The left member of (2) is then at most equal to

$$(4) \quad \left| \sum_{k=1,l=1}^{p,q} \right| + \left| \sum_{k=p+1,l=q+1}^{m,n} \right| + \left| \sum_{k=p+1,l=1}^{m,q} \right| + \left| \sum_{k=1,l=q+1}^{p,n} \right|,$$

in which the expression operated on by $\sum$ is in each case $a_{mnkl}x_{kl}$. We consider the four terms of (4) seriatim.

Let D (>0) be at least equal to the largest of the quantities

$$|x_{kl}|, \quad (k = 1, 2, \cdots, p;\ l = 1, 2, \cdots, q).$$

By (1) we have

$$\lim_{m,n\to\infty} a_{mnkl} = \lim_{m\to\infty} a'_{mk} \lim_{n\to\infty} a''_{nl} = 0.$$

Hence there exist numbers M_1, N_1 such that we have, for $m>M_1$, $n>N_1$,

$$|a_{mnkl}| < \epsilon/(4pqD), \qquad (k = 1, 2, \cdots, p;\ l = 1, 2, \cdots, q);$$

and the first term of (4) is at most equal to

$$\sum_{k=1,l=1}^{p,q} \epsilon |x_{kl}| /(4pqD) \leqq \epsilon/4$$

for $m>M_1$, $n>N_1$. By (3) and (1) the second term of (4) is less than or equal to

$$\sum_{k=p+1,\,l=q+1}^{m,n} |\,a_{mnkl}\,|\,\epsilon/(4K^2) = \frac{\epsilon}{4\,K^2} \sum_{k=p+1}^{m} |\,a'_{mk}\,| \sum_{l=q+1}^{n} |\,a''_{nl}\,| < \epsilon/4.$$

The third term of (4) is less than or equal to

$$\begin{aligned}\sum_{l=1}^{q} \left| \sum_{k=p+1}^{m} a_{mnkl}x_{kl} \right| &= \sum_{l=1}^{q} |\,a''_{nl}\,| \cdot \left| \sum_{k=p+1}^{m} a'_{mk}x_{kl} \right| \\ &\leqq \sum_{l=1}^{q} |\,a''_{nl}\,| \cdot \left[\left| \sum_{k=1}^{m} a'_{mk}x_{kl} \right| + \left| \sum_{k=1}^{p} a'_{mk}x_{kl} \right| \right] \\ &\leqq \sum_{l=1}^{q} |\,a''_{nl}\,| \cdot \left[B_l + D \sum_{k=1}^{p} |\,a'_{mk}\,| \right],\end{aligned}$$

where B_l (>0) denotes a bound for the A'-transform of the lth column of $\{x_{mn}\}$. Let B be at least equal to the largest of the bounds B_l, ($l=1, 2, \cdots, q$); then by (1) we infer the existence of numbers M_2, N_2 such that we have

$$|\,a'_{mk}\,| < B/(pD), \qquad (\text{for } m > M_2;\ k = 1, 2, \cdots, p);$$
$$|\,a''_{nl}\,| < \epsilon/(8qB), \qquad (\text{for } n > N_2;\ l = 1, 2, \cdots, q).$$

Thus the third term of (4) is less than $\epsilon/4$ for $m>M_2$, $n>N_2$. Similarly there exist numbers M_3, N_3 such that the fourth term of (4) is less than $\epsilon/4$ for $m>M_3$, $n>N_3$.

Hence if M be taken as the largest of M_1, M_2, M_3, and N as the largest of N_1, N_2, N_3, the inequality (2) will be satisfied. This completes the proof.

If we do not restrict ourselves to double A-transformations which are the products of simple A-transformations, the above proof can easily be modified to yield a second theorem, of which Theorem 1 is a particular case. For simplicity of statement let us first recall Robison's generalization of the Silverman-Toeplitz theorem.*

Necessary and sufficient conditions that a double A-transformation be regular **for the class of bounded sequences** *are*

$$\lim_{m,n\to\infty} a_{mnkl} = 0, \qquad (\textit{for each } k \textit{ and } l);$$

and

* Robison, loc. cit., p. 53.

$$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{mnkl} = 1;$$

$$\lim_{m,n\to\infty} \sum_{k=1}^{m} | a_{mnkl} | = 0, \qquad (\textit{for each } l);$$

$$\lim_{m,n\to\infty} \sum_{l=1}^{n} | a_{mnkl} | = 0, \qquad (\textit{for each } k);$$

$$\sum_{k=1,l=1}^{m,n} | a_{mnkl} | < K, \quad (K = \textit{constant}).$$

We then have the following theorem.

THEOREM 2. *If a double A-transformation is regular for the class of bounded sequences, it is also regular for the class of sequences* $\{x_{mn}\}$ *satisfying the following conditions*: (a) *each simple sequence* $\{x_{ml}\}$ *(l fixed) is carried by the transformation*

$$u_m^{nl} = \sum_{k=1}^{m} a_{mnkl} x_{kl}, \qquad (n \textit{ fixed}),$$

into a sequence $\{u_m^{nk}\}$ *whose elements are bounded by a constant* B_{nl} *such that, for each* l, $\lim_{n\to\infty} B_{nl} = 0$; (b) *each simple sequence,* $\{x_{kn}\}$ *(k fixed) is carried by the transformation*

$$v_n^{mk} = \sum_{l=1}^{n} a_{mnkl} x_{kl}, \qquad (m \textit{ fixed}),$$

into a sequence $\{v_n^{mk}\}$ *whose elements are bounded by a constant* C_{mk} *such that, for each* k, $\lim_{m\to\infty} C_{mk} = 0$.

3. *Application to Cesàro Summability of Double Series.* Moore* has defined summability (C, r) of a double series

$$\sum_{i=1,j=1}^{\infty,\infty} u_{ij} \tag{5}$$

in the following natural manner. Let us employ the notation

$$S_{mn} = \sum_{i=1,j=1}^{m,n} u_{ij}, \qquad S_{mn}^{(r)} = \sum_{i=1,j=1}^{m,n} A_{m-i+1}^{(r-1)} \cdot A_{n-j+1}^{(r-1)} S_{ij},$$

$$A_l^{(r)} = \frac{\Gamma(r+l)}{\Gamma(r+1)\Gamma(l)}, \quad A_{mn}^{(r)} = A_m^{(r)} \cdot A_n^{(r)}.$$

* Moore, loc. cit., Transactions of this Society, 1913, p. 74.

Then if as m and n become infinite simultaneously but independently the limit of $S_{mn}^{(r)}/A_{mn}^{(r)}$ exists, the series (5) is said to be summable (C, r). He gave a proof, valid for series of real terms only, of the following lemma.

MOORE'S CONSISTENCY LEMMA. *If the series* (5) *is summable* (C, r), *where* r *is zero or any positive integer, and if furthermore we have*

$$(6) \qquad \left| S_{mn}^{(r)}/A_{mn}^{(r)} \right| < C, \qquad (m, n = 1, 2, 3, \cdots),$$

where C *is a constant, then the series is summable* $(C, r+1)$ *to the same sum and in addition we have*

$$\left| S_{mn}^{(r+1)}/A_{mn}^{(r+1)} \right| < C, \quad (m, n = 1, 2, 3, \cdots).$$

Moore expressed doubt as to whether the condition (6) were necessary. We now propose to show that it is not necessary by applying the results of §2, and to obtain theorems valid for series of complex as well as of real terms. To this end let us recall Moore's equations (26) and (27):

$$(7) \qquad \begin{aligned} S_{mn}^{(r+1)} &= \sum_{i=1,\,j=1}^{m,\,n} S_{ij}^{(r)}, & A_{mn}^{(r+1)} &= \sum_{i=1,\,j=1}^{m,\,n} A_{ij}^{(r)}, \\ S_{mn}^{(r)} &= \Delta_{11} S_{m-1,n-1}^{(r+1)}, & A_{mn}^{(r)} &= \Delta_{11} A_{m-1,n-1}^{(r+1)}, \end{aligned}$$

where, in general,

$$\Delta_{11} C_{m-1,n-1} = C_{m,n} - C_{m,n-1} - C_{m-1,n} + C_{m-1,n-1}.$$

We are interested in determining conditions under which, when the sequence

$$(8) \qquad \left\{ S_{mn}^{(r)}/A_{mn}^{(r)} \right\} = \left\{ \Delta_{11} S_{mn}^{(r+1)}/\Delta_{11} A_{mn}^{(r+1)} \right\}$$

converges, the sequence

$$(9) \qquad \left\{ S_{mn}^{(r+1)}/A_{mn}^{(r+1)} \right\}$$

will converge to the same limit. By means of the relations (7) it is easily shown that the sequence (8) is carried over into the sequence (9) by the transformation whose general term, a_{mnkl} in the notation of §1, is

$$a_{mnkl} = \frac{A_{kl}^{(r)}}{\sum_{i=1,j=1}^{m,n} A_{ij}^{(r)}} = \frac{A_k^{(r)} \cdot A_l^{(r)}}{\sum_{i=1}^{m} A_i^{(r)} \sum_{j=1}^{n} A_j^{(r)}} = \frac{A_k^{(r)}}{A_m^{(r+1)}} \cdot \frac{A_l^{(r)}}{A_n^{(r+1)}}.$$

Each of these factors is the general term of the simple A-transformation by which the (C, r)-transform of a simple sequence is carried over into the $(C, r+1)$-transform of that sequence; this transformation is at once seen to be regular. Hence by Theorem 1, taking into account the fact that S_{ij} is the sum of partial sums for i rows or j columns of (5), we obtain the following consistency theorem.

THEOREM 3. *If the series* (5) *is summable* (C, r), *where r is zero or any positive integer, and if each row and column of* (5) *is finite* $(C, r+1)$, *then the series is summable* $(C, r+1)$ *to the same sum.*

In a similar manner we may establish the following more general consistency theorem.

THEOREM 4. *If the series* (5) *is summable* (C, r), *where r is zero or any positive integer, and if each row and column of* (5) *is finite* $(C, r+p)$ *for any positive integer p, then the series is summable* $(C, r+p)$ *to the same sum.*

Since a simple series finite $(C, r+p)$ is finite $(C, r+p+p')$ for any positive integer p', it follows that under the conditions stated in this theorem, the series (5) is also summable $(C, r+p+p')$ to the same sum.

If in defining Cesàro summability of (5), Cesàro factors of different orders, say r and s, are associated with rows and columns,* we may by a similar discussion obtain the following result.

THEOREM 5. *If the series* (5) *is summable* (C, r, s), *where each of the integers r, s is* $\geqq 0$, *and if each row of* (5) *is finite* $(C, s+q)$ *and each column finite* $(C, r+p)$, *where each of the integers p, q is* $\geqq 0$, *then the series is summable* $(C, r+p, s+q)$.

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* See Merriman, loc. cit., Annals of Mathematics, 1927, p. 525.