USEFUL FUNCTIONS ASSOCIATED WITH RATIONAL CUBIC CURVES*

BY CLIFFORD BELL

1. *Introduction.* The usual method of determining the Plücker numbers for plane curves is generally laborious. However, in certain cases, by the use of special theorems, the numbers may be determined at once. Thus, the curve $x_1:x_2:x_3=f(t):\phi(t):\psi(t)$, where f, ϕ, ψ are polynomials of degree 3 in t and the coefficients of t^2 in each are zero, has one cusp.† Also any cusp or node at a vertex of the triangle of reference is easily recognized by the form of the equation. The purpose of this paper is to derive some of the properties of two functions that are useful in determining whether the rational cubic is nodal or cuspidal.

2. *The Cubic Circumscribed about the Triangle of Reference.* The parametric equations of the cubic are

$$\rho x_i = (\lambda - \lambda_i)(\lambda - \lambda_{i+2})(\lambda - s_i), \qquad (i = 1, 2, 3), \tag{1}$$

where λ_i, λ_{i+2}, s_i are the points of intersection of the cubic with the side x_i of the triangle, λ_i and λ_{i+2} being vertices. Let A represent the function

$$(\lambda_3 - s_1)(\lambda_1 - s_2)(\lambda_2 - s_3) - (\lambda_1 - s_1)(\lambda_2 - s_2)(\lambda_3 - s_3).$$

THEOREM 1. *The cubic* (1) *has a cusp at one of the vertices of the triangle of reference or is nodal when A vanishes.*

PROOF. The class of the cubic is given by the degree of λ in the equation of a tangent line to the curve from any point (x_1, x_2, x_3), after all common factors are eliminated. The tangent line is given by the equation

$$\sum_{i=1}^{3} x_i(a_i\lambda^4 + b_i\lambda^3 + c_i\lambda^2 + d_i\lambda + e_i) = 0, \tag{2}$$

where

$$a_i = -\lambda_i + \lambda_{i+2} + s_{i+2} - s_{i+1},$$
$$e_i = \lambda_{i+1}^2(\lambda_i\lambda_{i+2}s_{i+2} + \lambda_{i+2}s_{i+1}s_{i+2} - \lambda_i s_{i+1}s_{i+2} - \lambda_i\lambda_{i+2}s_{i+1}),$$

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† For the general theorem, see Hilton, *Plane Algebraic Curves*, 1920, p. 151.

and b_i, c_i, d_i are other functions of the same quantities. It is readily seen that not more than one of a_1, a_2, a_3 can be zero without all being zero. When $A = a_1 = 0$,

$$a_2 = 2\frac{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)}{(s_3 - 2\lambda_2 + \lambda_3)} \neq 0,$$

and

$$a_3 = 2\frac{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{(s_2 - 2\lambda_2 + \lambda_1)} \neq 0.$$

Also if $a_2 = 0$, then a_1 and a_3 are not zero; or if $a_3 = 0$, a_1 and a_2 are different from zero. Hence the cubic is nodal unless the coefficients of x_1, x_2, x_3 in (2) have a common factor.

Suppose the coefficients of x_1, x_2, x_3 have a common factor $\lambda + \alpha$. Make the transformation $\lambda = \lambda' - \alpha$ on λ. The equation of the tangent line to the curve from any point (x_1, x_2, x_3) is given by (2) with primes on all the letters in the coefficients of x_1, x_2, x_3. In particular,

$$e_i = \lambda'^2_{i+1}(\lambda'_i \lambda'_{i+2} s'_{i+2} + \lambda'_{i+2} s'_{i+1} s'_{i+2} - \lambda'_i s'_{i+1} s'_{i+2} - \lambda'_i \lambda'_{i+2} s'_{i+1}), \quad (i = 1, 2, 3),$$

where $\lambda'_i = \lambda_i + \alpha$, $s'_i = s_i + \alpha$. Under this transformation it is seen that

$$A' = (\lambda'_3 - s'_1)(\lambda'_1 - s'_2)(\lambda'_2 - s'_3) - (\lambda'_1 - s'_1)(\lambda'_2 - s'_2)(\lambda'_3 - s'_3) = A.$$

As $\lambda + \alpha$ is a factor of the old coefficients, λ' is a factor of the new ones. Therefore e'_1, e'_2, e'_3 are all zero. This happens when and only when $\lambda'_i = s'_i = s'_{i+1} = 0$, which gives a cusp at a vertex.

It should be noted in the application of this theorem that a cubic with a cusp at a vertex of the triangle of reference is easily recognized, for a common square factor appears in two of the three parametric equations of the curve. Then, when A vanishes, the cubic is nodal if it is not of the above type.

THEOREM 2. *If the cubic* (1) *is cuspidal, A is zero for those cubics for which the cusp falls at one of the vertices of the triangle of reference, and different from zero for all others.*

PROOF. Either a_1, a_2, a_3 vanish or the coefficients of x_1, x_2, x_3 have a common factor for a cuspidal cubic. If $a_1=a_2=a_3=0$, $A=2(s_1-s_2)(s_3-s_2)(s_3-s_1)\neq 0$. If the coefficients of x_1, x_2, x_3 have a common factor, $\lambda+\alpha$, make the transformation $\lambda=\lambda'-\alpha$. The resulting constant terms, e_1', e_2', e_3', are all zero. It is easily shown that $A'=A\neq 0$ unless $\lambda_1'=s_1'=s_2'=0$, $\lambda_2'=s_2'=s_3'=0$ or $\lambda_3'=s_1'=s_3'=0$, in which cases a cusp falls at a vertex of the triangle of reference and $A=0$. For these cases $\lambda_1=s_1=s_2=\alpha$, $\lambda_2=s_2=s_3=\alpha$, or $\lambda_3=s_1=s_3=\alpha$.

THEOREM 3. *A triangle of reference, inscribed to any nodal cubic, can always be chosen such that A vanishes.*

PROOF. By a suitable choice of coordinates any crunodal cubic can be put in the form* $x_1:x_2:x_3=\lambda(\lambda^2-1):(\lambda^2-1):\lambda^3$ and any acnodal cubic in the form $x_1:x_2:x_3=\lambda(\lambda^2+1):(\lambda^2+1):\lambda^3$. A transformation

$$\begin{aligned}\rho y_1 &= (39x_1 - 14x_2 - 24x_3)/15,\\ \rho y_2 &= (-17x_1 - 30x_2 + 24x_3)/7,\\ \rho y_3 &= (34x_1 - 21x_2 - 24x_3)/10\end{aligned}$$

puts the crunodal cubic in the form (1), where λ_1, λ_2, λ_3, s_1, s_2, s_3 are, respectively, 2, 3, $-7/5$, $1/3$, $-5/7$, $1/2$ and for which $A=0$. Likewise a transformation can be found for which the acnodal cubic goes into the form (1) with A vanishing.

3. *The Cubic Inscribed to the Triangle of Reference.* The parametric equations of the cubic are

$$\rho x_i = (t-t_i)^2(t-r_i), \qquad (i=1,2,3), \tag{3}$$

where t_i and r_i are the parameters of the points of contact and intersection, respectively, of the cubic with $x_i=0$. Let the function

$$(t_3-r_1)(t_1-r_2)(t_2-r_3) - (t_2-r_1)(t_3-r_2)(t_1-r_3)$$

be represented by B.

THEOREM 4. *The cubic given by equations* (3) *has a cusp on one of the lines $x_1=0$, $(i=1, 2, 3)$, or is nodal when B vanishes.*

* Durège, Mathematische Annalen, vol. 1 (1869), pp. 513–515.

The proofs for this theorem and the next two are very similar to the proofs of the first three and hence will not be given. The cases where a cusp falls on the lines $x_i=0$ may easily be recognized. Thus if any two of t_1, t_2, t_3 are equal, a cusp falls at a vertex of the triangle of reference, and if the corresponding pair from r_1, r_2, r_3 are equal, B vanishes. A cusp on the lines $x_1=0$ but not at a vertex may be found by noting that t_1, t_2, t_3 are the only possible parameters for such a cusp and that the tangent line through it is indeterminate. In this case also B may vanish.

THEOREM 5. *If the cubic* (3) *is cuspidal, B may or may not be zero when the cusp is on one of the lines $x_i=0$,* $(i=1, 2, 3)$, *and B is different from zero for all other cases.*

THEOREM 6. *A triangle of reference, circumscribed to any nodal cubic, can always be chosen such that B vanishes.*

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AN APPLICATION OF METRIC GEOMETRY TO DETERMINANTS*

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1. *Introduction.* A paper presented to the Accademia dei Lincei by B. Segre† is devoted to the following theorem, announced by H. W. Richmond:‡

If in a non-vanishing, symmetric determinant of order six, the six elements in the principal diagonal are all zero, and the complementary minors of five of these elements are also zero, then the complementary minor of the remaining element must be zero.

Segre states that the analogous theorem for determinants of the second§ and the fourth orders may be immediately verified, and the object of his investigation is to ascertain if analogous theorems are valid for determinants of other orders. He shows

* Presented to the Society, September 9, 1931.

† *Intorno ad una proprietà dei determinanti simmetrici del 6° ordine,* Atti dei Lincei, (6), vol. 2 (1925), p. 539.

‡ *On the property of a double-six of lines, and its meaning in hypergeometry,* Proceedings of the Cambridge Philosophical Society, vol. 14 (1908), p. 475. The statement of the theorem given in this paper contains no explicit hypothesis relative to the non-vanishing of the determinant.

§ The theorem is, of course, trivial for second-order determinants.