

MAPS OF CERTAIN CYCLIC INVOLUTIONS ON TWO-DIMENSIONAL CARRIERS

BY W. R. HUTCHERSON

1. *Introduction.* The following paper derives the fundamental properties of the involutions on an algebraic surface which have but a finite number of invariant points. Except for a few particular cases, they cannot be regarded as subcases of those having a curve of invariant points; they require one more equation for their definition, analogous to the singular correspondences on algebraic curves. They exist only on surfaces having particular moduli.

2. *Discussion of I_n .* Consider two surfaces $F(x)=0$ and $\Phi(x')=0$ with the property that any point P on $F(x)=0$ uniquely fixes a point P' on $\Phi(x')=0$ and, conversely, the point P' fixes n points $P_1 \equiv P, P_2, \dots, P_n$ on F . There is thus set up an $(n, 1)$ correspondence between the points of $F=0$ and $\Phi=0$. Now any one of the n points P_1, \dots, P_n on $F=0$ definitely determines the whole group of n points to which it belongs. Hence, it will be said that F contains an involution I_n of order n , and that this I_n belongs to $\Phi(x')=0$.

There are two kinds of involutions; F may contain one or more curves, each point of which contains two or more coincidences of these n points $P_1, \dots, P_n, P_i = P_k$. Such curves are called *curves of coincidences*. The surface $\Phi(x')=0$ then contains a locus of branch points in $(1, 1)$ correspondence with the curve of coincidences on F . The other kind of involution is such that F has only a finite number of coincident points. Thus, $\Phi(x')$ has in this case exactly the same number of branch points.

If $\Phi(x')=0$ is a rational surface, or a plane, I_n is said to be *rational*. If $F(x)=0$ is rational, $\Phi(x')=0$ must be rational.* The converse is not true.

In this paper only I_n on $F(x)=0$ with a finite number of coincident points will be considered. Such an I_n can be gener-

* Castelnuovo, *Mathematische Annalen*, vol. 44 (1894), pp. 125-155.

ated by a group of birational transformations of the surface $F(x)=0$ into itself. The involution is cyclic, abelian, etc. according as it is generated by a cyclic, abelian, etc. group of transformations. This group is, of course, cyclic if n is prime. However, it may or may not be birational (Cremonian) for the whole space in which $F(x)=0$ lies.*

3. *Space of r Dimensions.* In a space S_r of r dimensions, it is possible to reduce $F(x)=0$ and $\Phi(x')=0$ to their normal forms. A surface is said to be *normal* in a linear S_r when it can not be obtained as the projection of a surface of the same order from a space S_m , $m > r$.

Call T the cyclic transformation which generates I_n (where n is a prime) on $F(x)=0$, and P a point of coincidence. Let C be any curve on $F(x)=0$, through P . The image of C under T is another curve on F through P . The point P is called a *perfect point of coincidence* if C and every image of C touches the same line at P for every tangent to F at P . Otherwise P is *non-perfect*. The corresponding branch point on $\Phi(x')=0$ is said to be perfect or non-perfect according as P is a perfect or non-perfect point of coincidence.

When F is reduced to its normal form, the operations of I_p can be represented by a collineation of period p , under which F is invariant in S_r . Since $p < r$, there exist not more than p spaces of invariant points. The form of each transformation is $x_i = \theta^i x'_i$, where $\theta^p = 1$.

4. *Discussion of I_2 .* THEOREM 1. *If $n=2$, every point P of coincidence on $F(x)=0$ must be a perfect point.*

PROOF. It has been proved that, given an I_p , it can always be represented by a collineation in S_R , having p axes, or spaces of invariant points, only one of which meets the surface F which contains the involution.

When $p=2$ there are then only two axes. The transformation T may be written

$$\begin{aligned} X'_1 : X'_2 : \cdots : X'_{r+1} : X'_{r+2} : \cdots : X'_{r+s+2} \\ = X_1 : X_2 : \cdots : X_{r+1} : \epsilon X_{r+2} : \cdots : \epsilon X_{r+s+2}, \end{aligned}$$

where $\epsilon = -1$.

* F. Enriques, *Bologna Rendiconti*, (2), vol. 14 (1910), pp. 71-75.

Consider an invariant point A on a surface F in S_R , where $R=r+s+1$. Call the hyperplanes

$$\sum_{i=1}^{r+1} s_i x'_i = 0, \Sigma^{(0)}, \text{ and } \sum_{k=r+2}^{r+s+2} r_k x'_k = 0, \Sigma^{(1)}.$$

Now $\Sigma^{(0)}$ contains one axis $S^{(1)}$ and $\Sigma^{(1)}$ the other $S^{(0)}$. Require $\Sigma^{(0)}$ to pass through point A in axis $S^{(0)}$. It cuts F in a system of invariant curves $|C|$, on each of which point A is the only coincident point. The tangents to $|C|$ at point A on F all lie in one tangent plane. $\Sigma^{(0)}$ contains this plane. There are two invariant directions at A in this plane or all are invariant. Call two invariant directions a_1 and a_2 ; then a_1 must have another invariant point A_1 on it. The point A_1 must lie in the other axis $S^{(1)}$ for A is the only point of $S^{(0)}$ in the tangent plane. Likewise a_2 must have another invariant point A_2 . It must also lie in $S^{(1)}$. A linear combination of the coordinates of A_1 and A_2 gives ∞ points on line A_1A_2 , each of which is a coincident point (invariant point). Line A_1A_2 lies in axis $S^{(1)}$. Hence, by joining A and every point on A_1A_2 , we obtain ∞ invariant directions. Therefore point A is a perfect point. This holds for a rational and an irrational surface.

Two illustrations of I_2 are given. The first one uses the quadratic transformation $x_i = 1/x'_i$ in S_2 with four fixed points $(\pm 1, \pm 1, 1)$. Nets* of cubic curves are mapped on the Cayley cubic surface with four nodes.

The second illustration maps a surface of Enriques of order six upon a double plane. The Cremona involution in S_3 , $y_i = 1/y'_i$, has eight fixed points $(1, \pm 1, \pm 1, \pm 1)$ on the surface F .

Several theorems follow, the proofs of which are omitted. †

THEOREM 2. Consider on an algebraic surface F a point A , which is non-perfect in a cyclic involution of third order under

* A. Emch, *On the invariant net of cubics in the Steinerian transformation*, this Bulletin, vol. 24 (1918), pp. 327–330. *On plane algebraic curves which are invariant under a quadratic Cremona transformation*, Tôhoku Mathematical Journal, vol. 21(1922), pp. 310–326.

† Earlier proofs of these theorems were also given by Godeaux. See Sisam, *Involutions of irrational surfaces*, Bulletin of the National Research Council, No. 63, vol. 14 (1928), pp. 295–309. Godeaux, *Brussels Bulletin*, 1927, pp. 524–543.

which F is invariant; it follows that in the domain of the first order of A , there are two distinct fixed points. These points are perfect coincidences of the involution.

THEOREM 3. *Consider a cyclic involution of prime order, having only a finite number of fixed points, belonging to an algebraic surface F , and possessing a non-perfect fixed point adjacent to which are two perfect fixed points; then this involution is of order three.*

A map of I_3 belonging to an irrational quintic surface upon an irrational surface in S_7 , has been discussed. There are two perfect and five non-perfect coincidences.

5. *Discussion of I_5 belonging to F_3 in S_3 .* Consider the surface

$$F_3(x_1x_2x_3x_4) \equiv ax_1^2x_3 + bx_2^2x_1 + cx_3^2x_4 + dx_4^2x_2 = 0,$$

in S_3 , invariant under the cyclic collineation T of order five

$$x'_1 : x'_2 : x'_3 : x'_4 = x_1 : \epsilon^1 x_2 : \epsilon^2 x_3 : \epsilon^3 x_4, \quad (\epsilon^5 = 1).$$

There are four invariant axes $S^{(0)}, S^{(1)}, S^{(2)}, S^{(3)}$ each consisting of a point: $P_1 \equiv (1, 0, 0, 0)$, $P_2 \equiv (0, 1, 0, 0)$, $P_3 \equiv (0, 0, 1, 0)$, and $P_4 \equiv (0, 0, 0, 1)$. Each lies on the surface F , and since these are the only possible invariant axes, the surface F has only four points of coincidence.

Consider a curve C , not transformed into itself by T , and passing through P_1 . Take a plane $x_3 + \lambda x_4 = 0$ of the pencil passing through P_1 and P_2 , tangent to C . This plane $x_3 + \epsilon \lambda x_4 = 0$ by T and hence is non-invariant. The curve cut out on F by $x_3 + \lambda x_4 = 0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Hence, the two curves do not touch each other at P_1 . Since C was a variable curve through P_1 satisfying the non-invariant property, it follows that P_1 is a non-perfect coincidence point. A similar argument shows that P_2, P_3 , and P_4 are also non-perfect coincidence points. The following theorem is proved.

THEOREM 4. *The I_5 belonging to F_3 in S_3 has four non-perfect points of coincidence.*

Consider the complete system cut out on F by the quintic surfaces. Let $|A|$ be the system. Its dimension is 55, its genus is 31, and the number of variable intersections of two members of the system is 75. A curve A of this system is not in general transformed into itself by T . There are, however, five partial systems

in $|A|$ which are transformed into themselves. Call these $|A_1|$, $|A_2|$, $|A_3|$, $|A_4|$, and $|A_5|$. By use of $|A_1|$, we find

$$\begin{aligned} & a_{11111}x_1^5 + a_{22222}x_2^5 + a_{33333}x_3^5 + a_{44444}x_4^5 + a_{12444}x_1x_2x_4^3 \\ & + a_{12223}x_1x_2^3x_3 + a_{13344}x_1x_3^2x_4^2 + a_{11233}x_1^2x_2x_3^2 \\ & + a_{11224}x_1^2x_2^2x_4 + a_{11134}x_1^3x_3x_4 + a_{23334}x_2x_3^3x_4 \\ & + a_{22344}x_2^2x_3x_4^2 = 0. \end{aligned}$$

We refer the curves A_1 projectively to the hyperplanes of a linear space of eleven dimensions. We obtain a surface Φ , of order 15, with hyperplane sections of genus 7, as the image of I_5 .

The equations of the transformation for mapping I_5 upon Φ in S_{11} are

$$\begin{aligned} \rho X_1 &= x_1^5, & \rho X_4 &= x_4^5, & \rho X_7 &= x_1x_3^2x_4^2, & \rho X_{10} &= x_1^3x_3x_4, \\ \rho X_2 &= x_2^5, & \rho X_5 &= x_1x_2x_4^3, & \rho X_8 &= x_1^2x_2x_3^2, & \rho X_{11} &= x_2x_3^3x_4, \\ \rho X_3 &= x_3^5, & \rho X_6 &= x_1x_2^3x_3, & \rho X_9 &= x_1^2x_2^2x_4, & \rho X_{12} &= x_2^2x_3x_4^2. \end{aligned}$$

By eliminating ρ , x_1 , x_2 , x_3 , x_4 from these twelve equations and from $F_3(x_1x_2x_3x_4) = 0$, we get as the nine equations defining the surface Φ ,

$$\begin{aligned} \left\| \begin{array}{ccc} X_4 & X_5 & X_{12} \\ X_5 & X_9 & X_6 \end{array} \right\| &= 0, & \left\| \begin{array}{ccc} X_2 & X_9 & X_{12} \\ X_6 & X_{10} & X_7 \end{array} \right\| &= 0, & \left\| \begin{array}{ccc} X_1 & X_{10} & X_8 \\ X_{10} & X_7 & X_{11} \end{array} \right\| &= 0, \\ & & \left\| \begin{array}{ccc} X_3 & X_{11} & X_7 \\ X_{11} & X_{12} & X_5 \end{array} \right\| &= 0, \end{aligned}$$

and $aX_8 + bX_6 + cX_{11} + dX_{12} = 0$. Designate by P'_1 the branch point of Φ corresponding to the point P_1 on F . The coordinates of P'_1 are all zero except X_1 .

The curves A_1 on F pass through P_1 if $a_{11111} = 0$. The tangent plane at P_1 to F is $x_3 = 0$. Now, the system of quintic surfaces passing through P_1 cuts $x_3 = 0$ in the curves $x_3 = 0$, $a_{22222}x_2^5 + a_{44444}x_4^5 + a_{12444}x_1x_2x_4^3 + a_{11224}x_1^2x_2^2x_4 = 0$. For general values of the constants this is a quintic curve with a triple point at P_1 , two branches being tangent to the line $x_2 = x_3 = 0$ and one to the line $x_3 = x_4 = 0$. When $a_{12444} = a_{11224} = 0$, the plane quintic curve breaks up into five lines through P_1 . These are all distinct except when either $a_{22222} = 0$ or $a_{44444} = 0$, when they coincide with $x_3 = x_4 = 0$ or $x_2 = x_3 = 0$, respectively. Since P_1 is non-

perfect, the $|A_1|$ through P_1 must have five distinct branches, unless each branch touches one of the two invariant directions.

In the plane $x_3 = 0$ the involution I_5 is generated by the homography T_1 , which is $x'_1 : x'_2 : x'_4 = x_1 : \epsilon x_2 : \epsilon^3 x_4$. By use of the plane quadratic transformation S , which is $x_1 : x_2 : x_4 = z_1^2 : z_1 z_2 : z_2 z_4$ and its inverse $z_1 : z_2 : z_4 = x_1 x_2 : x_2^2 : x_1 x_4$, we can investigate the character of the adjacent invariant points along the two invariant directions at P_1 . By the application of $ST_1 S^{-1} \equiv T'_1$,

$$\begin{aligned} (z_1, z_2, z_4)^{S^{-1}} &\sim (x_1 x_2, x_2, x_1 x_4)^{T_1} \sim (\epsilon x_1 x_2, \epsilon^2 x_2^2, \epsilon^3 x_1 x_4), \\ \text{or} \quad (x_1 x_2, \epsilon x_2^2, \epsilon^2 x_1 x_4)^S &\sim (z_1, \epsilon z_2, \epsilon^2 z_4). \end{aligned}$$

Thus the new transformation T'_1 is $x'_1 : x'_2 : x'_4 = x_1 : \epsilon x_2 : \epsilon^2 x_4$. The invariant point adjacent to P_1 along the line $x_3 = x_4 = 0$ is still a non-perfect coincidence point. Investigate the next point by use of $ST'_1 S^{-1} \equiv T''_1(z_1, z_2, z_4) \sim (z_1, \epsilon z_2, \epsilon z_4)$. This point is a perfect point of coincidence.

By use of another quadratic transformation R , namely $x_1 : x_2 : x_4 = z_1^2 : z_2 z_4 : z_1 z_4$ and its inverse $z_1 : z_2 : z_4 = x_1 x_4 : x_1 x_2 : x_4^2$, the adjacent point to P_1 along $x_2 = x_3 = 0$ can be investigated. Applying $RT_1 R^{-1}$ as above, we have $(z_1, z_2, z_4) \sim (\epsilon^2 z_1, z_2, z_4)$. Hence we get a perfect coincidence point. The following theorem is proved.

THEOREM 5. *The non-perfect coincidence point P_1 on F has one adjacent perfect point along the line $x_2 = x_3 = 0$, a non-perfect one along the line $x_3 = x_4 = 0$, with a perfect one adjacent to this.*

The tangent plane to F at $P_2(0, 1, 0, 0)$ is $x_1 = 0$. The homography T_2 in $x_1 = 0$ is $x'_2 : x'_3 : x'_4 = x_2 : \epsilon x_3 : \epsilon^2 x_4$. Apply $ST_2 S^{-1}$ and proceed as above. We find $x'_2 : x'_3 : x'_4 = x_2 : \epsilon x_3 : \epsilon x_4$. Hence, the adjacent point along $x_1 = x_3 = 0$ is perfect. By use of $RT_2 R^{-1}$ we obtain $(z_2, z_3, z_4) \sim (z_2, z_3, \epsilon z_4)$. This indicates a non-perfect point. By use of $RT'_2 R^{-1}$, we get $x'_2 : x'_3 : x'_4 = \epsilon^3 x_2 : x_3 : x_4$. This gives a perfect point. Hence we may state the following theorem.

THEOREM 6. *The non-perfect coincidence point P_2 on F has one adjacent point along the line $x_1 = x_4 = 0$, a non-perfect adjacent one along $x_1 = x_3 = 0$, with a perfect one adjacent to this.*

The point $P_3(0, 0, 1, 0)$ has $x_4 = 0$ for its tangent plane. The homography becomes, in this tangent plane, $T_3 : x'_1 : x'_2 : x'_3$

$= x_1 : \epsilon x_2 : \epsilon^2 x_3$. Introduce two quadratic transformations for use in discovering the nature of $P_3(0, 0, 1, 0)$. Calling the first one U and the second V , we have

$$\begin{aligned} U: & \quad y_1 : y_2 : y_3 = w_2 w_3 : w_1 w_2 : w_3^2, \\ U^{-1}: & \quad w_1 : w_2 : w_3 = y_2 y_3 : y_1^2 : y_1 y_3, \\ V: & \quad y_1 : y_2 : y_3 = w_1 w_2 : w_1 w_3 : w_2^2, \\ V^{-1}: & \quad w_1 : w_2 : w_3 = y_1^2 : y_1 y_3 : y_2 y_3. \end{aligned}$$

The adjacent points along the line $x_4 = x_2 = 0$ compel the use of $UT_3U^{-1} \equiv T_3'$, then $VT_3'V^{-1}$ or T_3'' , and then $UT_3''U^{-1}$ before a perfect point is found. We have $(w_1, w_2, w_3) \sim (\epsilon_3 w_1, w_2, \epsilon^2 w_3)$. This point is non-perfect. Consider $(w_1, w_2, w_3) \sim (\epsilon^3 w_1, w_2, \epsilon^2 w_3)$. This also is non-perfect.

Consider $(w_1, w_2, w_3) \sim (\epsilon^2 w_1, \epsilon^2 w_2, \epsilon^2 w_3)$. This is a perfect point in the neighborhood of P_3 of the third order.

Now consider the possibilities along $x_4 = x_1 = 0$, an invariant direction. Consider $(w_1, w_2, w_3) \sim (w_1, \epsilon^2 w_2, \epsilon^3 w_3)$. This is a non-perfect point. We have $(w_1, w_2, w_3) \sim (\epsilon^2 w_1, \epsilon^2 w_2, \epsilon^3 w_3)$. Hence we may state the following theorem.

THEOREM 7. *The non-perfect coincidence point P_3 on F has no adjacent perfect point of coincidence. There is one perfect point in the domain of the second order of P_3 and another in the domain of the third order of P_3 .*

The point $P_4(0, 0, 0, 1)$ has the plane $x_2 = 0$ for its tangent plane. The homography T in this plane becomes $T_4: x_1' : x_3' : x_4' = x_1 : \epsilon^2 x_3 : \epsilon^3 x_4$. Consider the direction $x_3 = x_2 = 0$ at P_4 . We have $(w_1, w_3, w_4) \sim (w_1, w_3, \epsilon^3 w_4)$. Hence, the adjacent point is perfect along $x_2 = x_3 = 0$. Now $(w_1, w_3, w_4) \sim (w_1, \epsilon^3 w_3, w_4)$ and $(w_1, w_3, w_4) \sim (w_1, w_3, \epsilon^3 w_4)$. Hence we have the following theorems.

THEOREM 8. *The non-perfect coincidence point P_4 has an adjacent perfect point along the line $x_2 = x_3 = 0$, a non-perfect one along the line $x_1 = x_2 = 0$, with a perfect one adjacent to this.*

THEOREM 9. *The system of invariant curves cut upon F by surfaces of degree lower than five all pass through the four coincidence points along the invariant directions. The number of branches through each point is less than five.*

CORNELL UNIVERSITY