REPRESENTATION OF A GROUP AS A TRANSITIVE PERMUTATION GROUP

BY G. A. MILLER

Let $G$ be any group of finite order $g$ and let $H$ be any subgroup of order $h$ contained in $G$. If the operators of $G$ are separated into right or into left augmented co-sets with respect to $H$ and if these $g/h = n$ co-sets are then multiplied successively on the right or on the left respectively by the various operators of $G$, they will be permuted as units according to a transitive permutation group $T$ which is simply isomorphic with the quotient group of $G$ with respect to the largest invariant subgroup of $G$ which appears in $T$, if $H$ is not itself invariant under $G$. If $H$ is invariant under $G$, then $T$ will be a regular group which is simply isomorphic with $G/H$. The case when $H$ is non-invariant under $G$ and does not involve any invariant subgroup of $G$ besides the identity is especially important since $T$ is then simply isomorphic with $G$, as was pointed out for right co-sets by W. Dyck in 1883.

If $K$ is any subgroup of $G$ which has operators in each of the co-sets of $G$ with respect to $H$ and if $K_0$ is the cross-cut of $H$ and $K$, then $K_0$ may be invariant under $K$ or it may involve an invariant subgroup under $K$. If one of these conditions is satisfied, $H$ must involve a subgroup which is invariant under $G$ and includes this invariant subgroup under $K$. This follows directly from the facts that this invariant subgroup is transformed into all of its conjugates under $G$ by operators of $H$ and that a complete set of conjugate subgroups always generates an invariant subgroup if it does not generate the entire group. We have then the following result.

**Theorem 1.** If a group $G$ is separated into co-sets with respect to a subgroup $H$ and if another subgroup $K$ has operators in each of these co-sets, then the largest invariant subgroup under $K$ which appears in the cross-cut of $H$ and $K$ is contained in an invariant subgroup of $G$ which is found in $H$.

In particular, when $T$ is simply isomorphic with $G$, then the largest invariant subgroup under $K$ which appears in the cross-cut of $H$ and $K$ is the identity.
A necessary and sufficient condition that a subgroup of $G$ corresponds to a transitive subgroup of degree $n$ in $T$ is that it has operators in each of the co-sets of $G$ with respect to $H$. In the case when $T$ is simply isomorphic with $G$ this transitive subgroup of degree $n$ must also be simply isomorphic with the corresponding subgroup of $G$. We have then the following result.

**Theorem 2.** When $G$ is represented as a simply isomorphic transitive permutation group of degree $n$ with respect to a subgroup $H$, then a necessary and sufficient condition that a given subgroup of $G$ corresponds to a simply isomorphic transitive subgroup of degree $n$ is that operators of this subgroup of $G$ appear in each of the co-sets of $G$ with respect to $H$.

When $T$ is not simply isomorphic with $G$, a subgroup of $G$ is not necessarily simply isomorphic with the corresponding subgroup of $T$. A necessary and sufficient condition that such a simple isomorphism exists is that this subgroup of $G$ has only the identity in common with the largest invariant subgroup of $G$ which is found in $H$. When $H$ is invariant under $G$, then the regular group $T$ will correspond to every subgroup of $G$ whose operators are distributed among all of the co-sets of $G$ with respect to $H$. Subgroups of $G$ whose operators are not thus distributed will correspond to regular constituents of the subgroups of $T$. A necessary and sufficient condition that a subgroup of $G$ corresponds either to a regular subgroup of $T$ or to a regular constituent of a subgroup of $T$ is that its cross-cut with $H$ is invariant under this subgroup.

Suppose that $G$ involves a subgroup $H$ whose operators are distributed among all except one of the co-sets of $G$ with respect to $H$ and that $T$ is simply isomorphic with $G$. The subgroup $H$ must correspond to a simply isomorphic subgroup of $T$ which is of degree $n-1$ and transitive on these $n-1$ letters. This subgroup must therefore appear in a conjugate of the subgroup of $T$ which corresponds to $H$. It therefore results that $T$ is multiply transitive. Moreover, when $T$ is multiply transitive, $G$ must contain such a subgroup.

**Theorem 3.** A necessary and sufficient condition that a group $G$ appears as a multiply transitive group when it is represented as a transitive permutation group with respect to a subgroup $H$ is that at least one subgroup of $G$ has its operators distributed among all except one of the co-sets of $G$ with respect to $H$. 

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From this theorem it results directly that a necessary and sufficient condition that a group is $r$-fold transitive when it is represented as a simply isomorphic transitive group with respect to a subgroup $H$ is that one can find $r - 1$ successive subgroups $H_1, H_2, \ldots, H_{r-1}$ each of which after the first is contained in the preceding and has its operators distributed among all except one of the co-sets of $G$ with respect to $H$ in which the operators of the preceding subgroup are found, the operators of $H_1$ appearing in all these co-sets except one. It may be noted that the operators of all of these $H$'s have the property that their products into any co-set which contains no operator of the corresponding $H$ appear in this co-set.

Suppose that all the operators of $G$ are distributed successively with respect to a subgroup $H$ of $G$ both into right and also into left co-sets. It is easy to verify that every such right co-set is identical with some left co-set of $G$ with respect to a conjugate of $H$ and that the totality of the right co-sets of $H$ includes left co-sets of $G$ with respect to all the conjugates of $H$ under $G$. Similarly, the totality of the left co-sets with respect to $H$ includes right co-sets with respect to all the conjugates of $H$ under $G$. This results directly from the theorem that the multiplying operators of co-sets can always be so chosen that the totality of the right multipliers is identical with the totality of left multipliers. This proves the following theorem.

**Theorem 4.** The right co-sets of any group $G$ with respect to a given subgroup $H$ are composed of left co-sets of $G$ with respect to all the conjugates of $H$ under $G$, and vice versa.

If more than one right co-set of $G$ with respect to $H$ is equal to a left co-set with respect to the same conjugate of $H$, the number of such right co-sets is equal to the index of $H$ under the largest subgroup of $G$ in which $H$ is invariant. In particular, this number is an invariant of $G$. A necessary and sufficient condition that only one right co-set of $G$ with respect to $H$ is equal to a left co-set with respect to a given conjugate of $H$ is that $H$ is transformed into itself only by its own operators under $G$. This is also a necessary and sufficient condition that only one left co-set of $G$ with respect to $H$ is equal to a right co-set with respect to a given conjugate of $H$. The operators of $G$ which when multiplied on the right into a given right co-set of $G$ with respect to $H$ have all their products in this co-set constitute the
conjugate of \( H \) in the equivalent left co-set, and vice versa.

A necessary and sufficient condition that \( T \) is simply transitive is that at least one conjugate of \( H \) under \( G \) has its operators distributed among less than \( n - 1 \) co-sets of \( G \) with respect to \( H \). In particular, when these operators are distributed among \( n - 2 \) such co-sets \( n \) must be even and \( T \) must involve a system of imprimitivity composed of \( n/2 \) sets of letters. This is also a necessary and sufficient condition that \( H \) is invariant under a subgroup of \( G \) whose order is exactly \( 2h \). If \( G \) is simply isomorphic with \( T \) and a subgroup of \( G \) has its operators distributed among \( n - 2 \) of the co-sets of \( G \) with respect to \( H \), its order cannot exceed \( 2h \), and when it has this order it must involve a subgroup of index 2 which is conjugate with \( H \) under \( G \). Moreover, \( H \) corresponds to a transitive subgroup of degree \( n - 2 \) in \( T \).

The University of Illinois

A NOTE ON TRANSFINITE ORDINALS

BY BEN DUSHNIK

In a supplementary note to an article of theirs,* Alexandroff and Urysohn demonstrated the following theorem.

If to every ordinal \( \alpha \) of the second class there corresponds an ordinal \( \mu(\alpha) \) such that \( \mu(\alpha) < \alpha \), then there exists a non-denumerable set of ordinals of the second class

\[
\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, \alpha_\omega, \ldots, \alpha_\lambda, \ldots
\]

such that

\[
\mu(\alpha_1) = \mu(\alpha_2) = \cdots = \mu(\alpha_\lambda) \cdots.
\]

The present note applies a different method to prove the following more general result.

THEOREM. Let \( \Omega_\delta \) be the smallest ordinal whose power is \( \aleph_\delta \), where \( \delta > 0 \) is a non-limiting ordinal. If to every transfinite ordinal \( \alpha < \Omega_\delta \) there corresponds an ordinal \( \mu(\alpha) \) such that \( \mu(\alpha) < \alpha \),

* Mémoire sur les espaces topologiques compacts, Verhandelingen of the Amsterdam Academy, (1), vol. 45, No. 1.