ON THE INTEGRATION OF UNBOUNDED FUNCTIONS*

BY W. M. WHYBURN

1. *Introduction.* The author has shown† that F. Riesz' treatment of integration‡ leads in every case to the Lebesgue integral. This demonstration makes possible a complete development of the theory of Lebesgue integration from the Riesz point of view.§ Such a development offers a number of advantages over the usual treatment and is especially desirable when one wishes to build a Lebesgue theory on a previous treatment of the Riemann integral. The purpose of the present paper is to emphasize further the importance of the Riesz point of view in a treatment of the general subject of integration by establishing additional relations between sequences of simple functions and functions that are summable in the senses of Lebesgue, Harnack, Denjoy, Denjoy-Khintchine-Young, and Young. The terminology and notation of Riesz' paper‖ are used.

2. *Preliminary Definitions and Theorems.* In this section we give a number of definitions and theorems which are well known but which are essential for the development of later theorems.

Simple function.¶ A function $\phi(x)$ is said to be a simple function on X: $a \leqq x \leqq b$, if there exist $n+1$ points: $x_0 = a < x_1 < x_2 < \cdots < x_n = b$, such that $\phi(x) \equiv \phi_i$, a constant, on $I_i: x_i < x < x_{i+1}$.

Null set. A set of points K is said to be of measure zero if for each $\epsilon > 0$, there exists an at most countably infinite set of intervals such that each point of K is an interior point of some one of these intervals and the sum of the lengths of the intervals

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* Presented to the Society, November 28, 1931.

† This Bulletin, vol. 37(1931), pp. 561–564.

‡ Acta Mathematica, vol. 42(1920), pp. 191–205.

§ This point of view is essentially that used in the ordinary treatment of the Riemann integral.

‖ Loc. cit. We work entirely in the real domain.

¶ These functions have also been called *horizontal* or *step functions*. In this connection see Ettlinger, American Journal of Mathematics, vol. 48 (1926), pp. 215–222.

is less than ϵ. Such a set of points is called a *null set*, and a property which holds everywhere on an interval X with the exception of points of a null set is said to hold *almost everywhere* on X, or is said to hold on X_0.

Integral of a simple function. By $\int_a^b \phi(x)dx$, where $\phi(x)$ is a simple function on X, we mean $\sum_{i=0}^{n-1}\phi_i\Delta_i$, where $\Delta_i = x_{i+1} - x_i$ is the length of I_i.

THEOREM 1. *If $f(x)$ is the limit function on X_0 of a uniformly bounded sequence $[\phi_n(x)]$ of simple functions, then*

$$\lim_{n\to\infty}\int_a^b \phi_n(x)dx$$

exists, and furthermore this limit is the same for all uniformly bounded sequences of simple functions which approach $f(x)$ on X_0.

Theorem 1 was proved by F. Riesz.* He used the limit whose existence is asserted in this theorem as a definition of $\int_a^b f(x)dx$. The author has established† the following theorem which shows that Riesz' integral is identical with the Lebesgue integral.

THEOREM 2. *A necessary and sufficient condition that a bounded function be measurable is that it be the limit function on X_0 of a uniformly bounded sequence of simple functions.*

Riesz established‡ the following theorem.

THEOREM 3. *A necessary and sufficient condition that a bounded function $f(x)$ be Riemann integrable on X is that there exist a sequence $[\phi_n(x)]$ of simple functions which approaches $f(x)$ uniformly almost everywhere on X, that is, uniformly in the neighborhoods of all points on X_0.*

Measure of a set of points. Let K be a set of points on the interval X and let $f(x) = 1$ at points of K and $f(x) = 0$ on $X - K$. If $\int_a^b f(x)dx$ exists§ the point set K is said to be measurable and its measure $m(K)$ is defined as the value of $\int_a^b f(x)dx$.

3. *Unbounded Functions.* We now consider functions which may be unbounded on their interval of definition $X: a \leqq x \leqq b$.

* Loc. cit., p. 196.

† W. M. Whyburn, loc. cit., pp. 561 and 564.

‡ Loc. cit., p. 204.

§ This is a Lebesgue integral but is defined by the Riesz method and hence uses the measure of null sets only in its definition.

THEOREM 4. *A necessary and sufficient condition that a function $f(x)$ be measurable on X is that there exist a sequence of simple functions that approaches $f(x)$ on X_0.*

PROOF. *Necessity.* Let $f(x)$ be measurable and let $F_i(x)=f(x)$ when $|f(x)|\leqq i$, $F_i(x)=i$ when $f(x)>i$, and $F_i(x)=-i$ when $|f(x)|<-i$. The bounded functions $F_i(x)$, $(i=1, 2, \cdots)$, are measurable and hence by Theorem 2 there exist sequences $[\phi_{in}(x)]$ of simple functions which approach these functions on X_0. By Egeroff's theorem* the sequence $[\phi_{in}(x)]$ approaches $F_i(x)$ uniformly on X except for a set of arbitrarily small measure. For each i choose an index N_i so that $|\phi_{iN_i}(x)-F_i(x)|<(1/2^i)$ except on a set E_i of measure less than $1/2^i$. Let $\phi_i(x)\equiv\phi_{iN_i}(x)$, $(i=1, 2, 3, \cdots)$. We prove that $[\phi_i(x)]$ is a sequence which approaches $f(x)$ on X_0. Let E be the subset of X on which $[\phi_i(x)]$ does not approach $f(x)$. If E is of positive measure, we can determine a number j such that the subset D of E on which $|f(x)|\leqq j$ is of positive measure c (otherwise E would consist of an at most countably infinite set of null sets and would therefore be of measure zero). Let j be chosen and then choose $n>j$ such that $1/2^n<c/4$. We have $|\phi_{n+r}(x)-F_{n+r}(x)|<1/(2^{n+r})$ except on a set of measure less than $1/(2^{n+r})$, $(r=1, 2, \cdots)$. At each point of D, $F_{n+r}(x)=f(x)$, and hence with the possible exception of a subset of measure less than $\sum_{r=1}^{\infty}1/(2^{n+r})<c/2$, we have $|\phi_{n+r}(x)-f(x)|<1/(2^{n+r})$, $(r=1, 2, \cdots)$, on D. Hence $\lim_{j\to\infty}\phi_j(x)=f(x)$ on a subset of D of measure $c/2$ which contradicts the assumption that this limit does not hold at any point of D. This completes the proof of the necessity.

Sufficiency. Let $[\phi_n(x)]$ be a sequence of simple functions which approaches $f(x)$ on X_0 and let M_1 and M_2, $M_1<M_2$, be any two constants. Let E be the subset of X on which $M_1\leqq f(x)\leqq M_2$. Let $F(x)=f(x)$ when $M_1-1\leqq f(x)\leqq M_2+1$, $F(x)=M_1-1$ when $f(x)<M_1-1$, $F(x)=M_2+1$ when $f(x)>M_2+1$. Let $\theta_i(x)=\phi_i(x)$ when $\phi_i(x)$ lies between M_1-1 and M_2+1, $\theta_i(x)=M_1-1$ when $\phi_i(x)\leqq M_1-1$, $\theta_i(x)=M_2+1$ when $\phi_i(x)\geqq M_2+1$. Now $[\theta_i(x)]$ is a uniformly bounded sequence of simple functions which approaches the bounded function $F(x)$ on X_0. Hence by Theorem 2, $F(x)$ is measurable on X and the set of points for which $M_1\leqq F(x)\leqq M_2$ is measurable. This set, how-

* See Hobson, *Functions of a Real Variable*, vol. 2, p. 140.

ever, is the set E and hence E is measurable. It follows from this that $f(x)$ is measurable on X.

Definition of property B. A sequence $[\phi_n(x)]$ of simple functions is said to have property B if for each $\epsilon>0$, there exists a number M_ϵ such that if $I(n, \epsilon)$ denotes the set of intervals on which $|\phi_n(x)|>M_\epsilon$, then $\int_{I(n,\epsilon)}|\phi_n(x)|dx<\epsilon$, $(n=1, 2, \cdots)$, where $\int_{I(n,\epsilon)}|\phi_n(x)|dx$ means the sum of the integrals of $|\phi_n(x)|$ on the separate intervals of $I(n, \epsilon)$.

THEOREM 5. *If* $[\phi_n(x)]$ *is a sequence of simple functions which has property B and which is such that* $\lim_{n\to\infty}\phi_n(x)$ *exists on* X_0, *then* $\lim_{n\to\infty}\int_a^b\phi_n(x)dx$ *exists.*

PROOF. Let $\epsilon=1$. We know that $|\int_a^b\phi_n(x)dx|\leqq M_1(b-a)+\int_{I(n,1)}|\phi_n(x)|dx<M_1(b-a)+1$ for all n. The set of numbers $K_n=\int_a^b\phi_n(x)dx$ is therefore bounded. Hence there is at least one number K which either is identical with infinitely many of the numbers K_n or is the limit of a subsequence of $[K_n]$. In either case we may pick out a sequence $[\theta_i(x)]$ of simple functions from the sequence $[\phi_n(x)]$ in such a way that $[\theta_i(x)]$ approaches the same limit function $f(x)$ as $[\phi_n(x)]$, has property B, and is such that $\lim_{i\to\infty}\theta_i(x)dx$ exists.

LEMMA. *If* $[h_n(x)]$ *and* $[g_n(x)]$ *are two sequences of simple functions which approach* $f(x)$ *on* X_0 *and which have property B, then if* $\lim_{n\to\infty}\int_a^bh_n(x)dx$ *and* $\lim_{n\to\infty}\int_a^bg_n(x)dx$ *exist, these limits are equal.*

PROOF OF LEMMA. Let us suppose that $\lim_{n\to\infty}\int_a^bh_n(x)dx=H$, $\lim_{n\to\infty}\int_a^bg_n(x)dx=G$, and let $\eta=7\epsilon>0$ be arbitrarily assigned. Choose N_1 so that for all $n>N_1$

$$(1)\qquad \left|\int_a^b h_n(x)dx-H\right|<\epsilon,\qquad \left|\int_a^b g_n(x)dx-G\right|<\epsilon.$$

Let M be the larger of the two numbers M_ϵ for $[h_n(x)]$ and $[g_n(x)]$ by property B. Let $J(n, \epsilon)$ and $J'(n, \epsilon)$, respectively, be the sets of intervals on which $|g_n(x)|>M$ and $|h_n(x)|>M$. Hence

$$\int_{J(n,\epsilon)}|g_n(x)|dx<\epsilon,\quad \int_{J'(n,\epsilon)}|h_n(x)|dx<\epsilon,\quad (n=1,2,\cdots).$$

Let $G_n(x)=g_n(x)$ when $|g_n(x)|\leqq M$, $G_n(x)=M$ when $g_n(x)>M$,

$G_n(x) = -M$ when $g_n(x) < -M$; and let $H_n(x) = h_n(x)$ when $|h_n(x)| \leqq M$, $H_n(x) = M$ when $h_n(x) > M$, $H_n(x) = -M$ when $h_n(x) < -M$. It follows immediately that $[\,|G_n(x) - H_n(x)|\,]$ is a sequence of uniformly bounded simple functions that approaches zero on X_0. By Theorems 1 and 2 we may choose an index N, $N > N_1$, such that for all $n > N$

$$\int_a^b |G_n(x) - H_n(x)|\, dx < \epsilon. \tag{4}$$

We have

$$\int_a^b G_n(x)dx - \int_{J(n,\epsilon)} [M + |g_n(x)|]dx \leqq \int_a^b g_n(x)dx$$

$$\leqq \int_a^b G_n(x)dx + \int_{J(n,\epsilon)} [M + |g_n(x)|]dx,$$

$$\int_a^b H_n(x)dx - \int_{J'(n,\epsilon)} [M + |h_n(x)|]dx \leqq \int_a^b h_n(x)dx$$

$$\leqq \int_a^b H_n(x)dx + \int_{J'(n,\epsilon)} [M + |h_n(x)|]dx.$$

Since $M < |g_n(x)|$ on $J(n, \epsilon)$ and $M < |h_n(x)|$ on $J'(n, \epsilon)$, we have

$$\int_a^b G_n(x)dx - 2\epsilon < \int_a^b g_n(x)dx < \int_a^b G_n(x)dx + 2\epsilon, \tag{2}$$

$$\int_a^b H_n(x)dx - 2\epsilon < \int_a^b h_n(x)dx < \int_a^b H_n(x)dx + 2\epsilon. \tag{3}$$

If we combine (2) and (3) and make use of (4), we obtain

$$\left| \int_a^b g_n(x)dx - \int_a^b h_n(x)dx \right| \tag{5}$$

$$< \left| \int_a^b [G_n(x) - H_n(x)]dx \right| + 4\epsilon < 5\epsilon, \text{ for } n > N.$$

A combination of (5) with (1) yields $|G - H| < 7\epsilon$, for all $n > N$. Since $\eta = 7\epsilon$ was arbitrarily chosen, it follows that $G = H$.

Proof of Theorem 5. If $\lim_{n\to\infty} \int_a^b \phi_n(x)dx$ is not K, we may pick a subsequence $[g_j(x)]$ from $[\phi_n(x)]$ which approaches $f(x)$

on X_0, has property B, and is such that $\lim_{j\to\infty}\int_a^b g_j(x)dx$ exists and is different from K. But by the lemma, $\int_a^b g_j(x)dx$ must approach the same limit as $\int_a^b \theta_i(x)dx$. It follows from this that $\lim_{n\to\infty}\int_a^b \phi_n(x)dx$ exists.

COROLLARY. *If* $[g_n(x)]$ *and* $[h_n(x)]$ *are any two sequences of simple functions that approach* $f(x)$ *on* X_0 *and that have property* B, *then* $\lim_{n\to\infty}\int_a^b g_n(x)dx$ *and* $\lim_{n\to\infty}\int_a^b h_n(x)dx$ *exist and these limits are equal.*

THEOREM 6. *A necessary and sufficient condition that a function* $f(x)$ *be summable* (*in the sense of Lebesgue*) *is that there exist a sequence* $[\phi_n(x)]$ *of simple functions which approaches* $f(x)$ *on* X_0 *and which has property* B.

Proof of sufficiency. This may be proved directly or we may obtain it as a consequence of a theorem due to de la Vallée Poussin.* We note that the absolute continuity of the integrals $\int\phi_n(x)dx$, $(n=1, 2, \cdots)$, is uniform,† since for a given $\epsilon>0$ we may choose $\delta=\epsilon/[2M_{\epsilon/2}]$, and δ is independent of n. It follows then from de la Vallée Poussin's theorem that $f(x)$ is summable on X.

Proof of necessity. Let $f(x)$ be summable on X and let X be subdivided into n equal parts by the points $x_1, x_2, \cdots, x_{n-1}$. Let $x_0=a$ and $x_n=b$. On $I_i: x_i\leqq x<x_{i+1}$, let $\phi_n(x)=\int_{x_i}^{x_{i+1}}f(x)dx/\Delta_i$, $(i=0, \cdots, n-1)$, where $\Delta_i=x_{i+1}-x_i$. It follows‡ from the absolute continuity of $\int f(x)dx$ that $\lim_{n\to\infty}\phi_n(x)=f(x)$ on X_0. It remains to show that $[\phi_n(x)]$ has property B. Let $g(x)=|f(x)|$, $h_n(x)=\int_{x_i}^{x_{i+1}}g(x)dx/\Delta_i$ on I_i, $(i=0, 1, \cdots, n-1)$. Let $\epsilon>0$ be assigned and let $\delta>0$ be chosen so that $\int_{(e)}g(t)dt<\epsilon$, when e is a measurable subset of X such that $m(e)\leqq\delta$. The existence of δ follows from the absolute continuity of $\int_a^x g(t)dt$. Let $g_j(x)=g(x)$ when $|g(x)|\leqq j$, $g_j(x)=j$ when $g(x)>j$. The integral $\int_a^b g_j(x)dx$ is an increasing function of j and has $\int_a^b g(x)dx$ as its limit as j becomes infinite. Choose J so that for all $j\geqq J$, $\int_a^b g(x)dx - \int_a^b g_j(x)dx<\epsilon$. Let k denote the subset of X on which $g(x)>J$ and let k_i denote the subset of k that lies on I_i. We have $\int_{(k)}g(x)dx=\sum_{i=0}^{n-1}\int_{(k_i)}g(x)dx<\epsilon$. Now

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* Transactions of this Society, vol. 16 (1915), p. 445.

† For definition, see de la Vallée Poussin, loc. cit. p. 445.

‡ See W. M. Whyburn, loc. cit., p. 562, for a detailed proof of this.

$$h_n(x) = [\int_{(I_i-k_i)} g(x)dx + \int_{(k_i)} g(x)dx]/\Delta_i$$
$$\leqq J + \int_{(k_i)} g(x)dx/\Delta_i, \text{ on } I_i.$$

Let $M_\epsilon = J + \epsilon/\delta$. In order that $h_n(x)$ be greater than M_ϵ on I_i it is necessary that $\int_{(k_i)} g(x)dx/\Delta_i$ be greater than ϵ/δ and hence Δ_i be less than $[\delta/\epsilon]\int_{(k_i)} g(x)dx$. From this it follows that the sum $I'(n, \epsilon)$ of intervals on which $h_n(x) > M_\epsilon$ is of measure less than $\sum_{i=0}^{n-1}[\delta/\epsilon]\int_{(k_i)} g(x)dx < \epsilon\delta/\epsilon \leqq \delta$. Hence $\int_{I'(n,\epsilon)} g(x)dx = \int_{I'(n,\epsilon)} h_n(x)dx < \epsilon$. Since $|\phi_n(x)| \leqq h_n(x)$, it follows that the set $I(n, \epsilon)$ of intervals on which $|\phi_n(x)| > M_\epsilon$ is a subset of $I'(n, \epsilon)$ and that $\int_{I(n,\epsilon)} |\phi_n(x)| dx$ is less than ϵ for all n. This proves that $[\phi_n(x)]$ has property B.

THEOREM 7. *If $f(x)$ is summable on X and if $[\phi_n(x)]$ is any sequence of simple functions which has property B and which approaches $f(x)$ on X_0, then* $\lim_{n\to\infty} \int_a^b \phi_n(x)dx$ *exists and is equal to $\int_a^b f(x)dx$.*

PROOF. This theorem follows immediately from Theorem 5 and a theorem of de la Vallée Poussin.*

A theorem of de la Vallée Poussin† enables us to state Theorem 6 in the following form.

THEOREM 8. *A necessary and sufficient condition that a function $f(x)$ be summable on X: $a \leqq x \leqq b$, is that there exist a sequence $[\phi_n(x)]$ of simple functions that approaches $f(x)$ on X_0 and is such that the absolute continuity of the indefinite integrals of these simple functions is uniform.*

We may now state and prove the following theorems which apply when the integrals are taken in senses that are more general than that of Lebesgue. Let $f(x)$ be a function on X: $a \leqq x \leqq b$, and let any definition of the integral of $f(x)$ be given which exists for $f(x)$ and which has the further property that if $J(a, b)$ denotes this integral and c, d, e are any three points of X, then $J(c, d)$, $J(c, e)$, $J(e, d)$ exist and $J(c, d) = J(c, e) + J(e, d)$.

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* Loc. cit., p. 446, Theorem 2. We note that the proof of Theorem V might be shortened by the use of de la Vallée Poussin's theorems. It seems desirable, however, to give this proof from the Riesz point of view in order that the theorem be available for use in a treatment of Lebesgue integration by the Riesz approach.

† Loc. cit., p. 450.

THEOREM 9. *Let E be an everywhere dense set of points on X and let M denote the subset of X which has the property that if p is any point of M, then* $\lim_{q\to p} J(p, q)/(q-p)$ *exists and equals $F(p)$, where q ranges on the set E. There exists a sequence $[\phi_n(x)]$ of simple functions such that this sequence converges to $F(x)$ at each point of M and $\int_a^b \phi_n(x)dx = J(a, b)$ for $n = 1, 2, \cdots$.*

PROOF. Let a method of subdivision of X be chosen in such a way that all of the subdivision points belong to E and the length of the maximum subdivision approaches zero as the number of these subdivisions becomes infinite. Let the points of subdivision, in order of increasing magnitude, for the nth stage of the subdivision be $x_0 = a, x_1, x_2, \cdots, x_n = b$. On I_i: $x_i \leqq x < x_{i+1}$, let $\phi_n(x) = J(x_i, x_{i+1})/(x_{i+1} - x_i)$, $(i = 0, 1, \cdots, n-1)$. Consider $[\phi_n(x)]$, $(n = 1, 2, \cdots)$. Let p be any point of M, and for each n let I_{p_n}: $p_n \leqq x < q_n$ be the subdivision that contains p. Now

$$\begin{aligned}\phi_n(p) &= J(p_n, q_n)/(q_n - p_n)\\ &= [J(p_n, p) + J(p, q_n)]/(q_n - p_n) = J(p_n, p)/(p - p_n)\\ &+ [J(p, q_n)/(q_n - p) - J(p_n, p)/(p - p_n)][(q_n - p)/(q_n - p_n)].\end{aligned}$$

We note that $|(q_n - p)/(q_n - p_n)| \leqq 1$ while its coefficient has zero for its limit (since both terms approach $F(p)$). Since $\lim_{n\to\infty} J(p_n, p)/(p - p_n)$ is $F(p)$, we have $\lim_{n\to\infty} \phi_n(p) = F(p)$. In case p is a point of subdivision,* we replace the indeterminate ratio in the above formula for $\phi_n(p)$ by the limit of this ratio (which is $F(p)$). Finally, we note that, for each n,

$$\int_a^b \phi_n(x)dx = \sum_{i=0}^{n-1} J(x_i, x_{i+1}) = J(a, b)$$

COROLLARY 1. *If E is identical with X, then M is the set of points at which $J(a, x)$ has a derivative with respect to x. In particular, if this derivative is equal to $f(x)$ at a point of M, then $[\phi_n(x)]$ converges to $f(x)$ at this point.*

COROLLARY 2. *If $f(x)$ is integrable in the sense of Harnack-Lebesgue, Denjoy, or Lebesgue, the conditions of Theorem 9 are satisfied if E is any everywhere dense set on X and M is X_0. Furthermore, $F(x) \equiv f(x)$ on X_0.*

COROLLARY 3. *If $f(x)$ is integrable in the Denjoy-Khintchine-Young sense and if X can be divided into a countable number of*

* We define $\phi_n(b) = \phi_n(x_{n-1})$.

measurable sets in such a way that the indefinite integrals (in the D-K-Y sense) are absolutely continuous on each of these, then the conditions of Theorem* 9 *are satisfied when $J(a, b)$ denotes the D-K-Y integral of $f(x)$, E is any everywhere dense set on X, M is X_0, and $F(x) \equiv f(x)$ on X_0.*

It is a very simple matter to extend the results of the present paper to cases where the interval of definition X is replaced by any measurable point set E on X. The definition of $f(x)$ is extended to points of $X-E$ by letting $f(x)$ be zero at such points. The integral of $f(x)$ on E is then described in terms of the integral of the extended function on X. One could define a simple function on a point set by saying that it is a function which takes on only a finite number of values on this set. This definition is not needed in the present adaptation.

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES

REFLECTIONS IN FUNCTION SPACE†

BY L. S. KENNISON

1. *Introduction.* The purpose of this paper is to point out an error,‡ giving the corrected form of the incorrect theorem referred to below as Delsarte's theorem, also to prove a generalization. However, the contribution made to the geometry of function space may be interesting to some on its own account.

We shall consider only functions of one or two variables defined on the interval (a, b) or the corresponding square. All such functions are to be bounded and integrable on the range of definition. We shall denote the continuous arguments on (a, b) by the letters x, s, t, u, written as subscripts or superscripts and shall imply integration on (a, b) with respect to any argument that occurs twice in the same term.

* See Khintchine, Comptes Rendus, vol. 152 (1916), p. 290.

† Presented to the Society, February 28, 1931.

‡ See §3 below.