COOLIDGE ON ALGEBRAIC CURVES

A Treatise on Algebraic Plane Curves. By Julian Lowell Coolidge. Oxford, Clarendon Press, 1931. xxiv+513 pp. $10.00.

Since the appearance of Salmon's treatise nearly three quarters of a century ago, the English language has contributed but two books on this subject, both of them elementary and conservative. The German quota was well up to date at the time of its appearance, but each representative is now out of date; it remained for Italy to enrich the literature by including the latest advances in a number of recent fundamental works.

It is therefore with a feeling of pride and of satisfaction that we welcome another volume which will be of great practical value not only to English-speaking readers, but to everyone who desires a comprehensive orientation in this fundamental theory. Although all the topics considered in the earlier books are discussed in the present volume, the plan is different from that of any other treatment. A great many, perhaps too many, points of view are introduced and discussed, though some are treated much more adequately than others. The expansion of the field during the last half-century has been simply appalling; it is impossible to prepare an all-inclusive view, and there is no criterion as to what should be retained, and what omitted.

The method is algebraic, and the most important concept is that of correspondence. A chapter on the algebra of polynomials precedes the subject proper, which begins with the conditions which determine a curve, a singular point, a tangent; and almost immediately one meets Noether's famous $Af+B\phi$ theorem. The pace is brisk, but each step is convincing. The next short chapter on asymptotes and singular points is less satisfactory. As the author clearly shows in a later chapter, the only adequate tool for studying either is by means of series, without any use of derivatives. In complicated cases the equations connecting derivatives soon become too cumbersome to be of any use.

A careful and detailed analysis of the real circuits of a curve is made, with elaborate drawings. These and a few simple ones on Newton's polygon are the only figures in the book. An attempt is made to give a mathematical meaning to the phrase "in general," but the attitude concerning it is very elastic. Sometimes it is set in quotation marks, sometimes italicized, but only infrequently is the content of the defining restriction explained. The case is not helped by using the expression "usually" without connotation.

The chapter on invariants includes an introduction to the Clebsch-Aronhold symbolism for forms in n variables. I believe it is far too brief and abrupt. Apart from its use in the transcendental treatment and for the formal part of invariants, I feel that its utility is questionable. In fact, the author's reference to it in his preface strengthens my point of view.

But still more debatable is the use of the partial derivative as to the number 1. I have examined the numerous cases in which it is used, and I cannot believe that it is justified in any one of them. In fact, the use of homogeneous and of non-homogeneous coordinates appears to be inconsistent. In the discussion of dual properties, point coordinates are treated as non-homogeneous and their corresponding line coordinates are homogeneous. Cartesian and projective systems are frequently confused.

The Cayley-Brill theory of correspondence is introduced early, and is used as a tool throughout the work. It seems to me unfortunate to use the name of Chasles to designate this fundamental theorem (Chasles-Cayley-Brill). Though historical details are not numerous, a footnote in this connection cites the essay in which it is shown that the names of two others are much more entitled to recognition than Chasles.

Chapter X of Book I, on metrical properties of curves, contains hardly a reference less than half a century old; many of these refer to books in which the theorems were already old at that time. This is the only chapter in the book that should have been omitted. Nothing in it is made use of later except a few elementary properties that could be explained in a few lines when needed.

It seems like reaching a new world when we strike Book II, on singular points. The treatments of quadratic inversion, expansion in series, and clustering singularities are excellent. The discussion of characteristic exponents and their inherent part in the given singularity is rapid but clear. The fundamental concept of satellite points is too briefly treated and rather too abstract. A schematic figure at this point would have spoken volumes. All this is done before adjoints are treated as such. The essence of the Brill-Noether theory, culminating with the Riemann-Roch theorem, can now proceed smoothly and naturally. A few repetitions appear. Thus the formula for the number of $(r+1)$-fold points of a g_n^r is given in exactly the same form on pages 133, 246, and 280.

A short chapter on abelian integrals precedes the concept of genus according to Weierstrass and that of moduli, the latter leading to the curve of lowest order with given genus and arbitrary moduli. The conditions under which further reduction can be made are not determined. After the general theorem of page 300, that of page 302 is curious. A lower order than that given is always possible under the same premises by projecting the canonical curve from any S_{p-3} meeting it in $p-3$ points into S_2. A very short chapter on special types of curves, including elliptic, hyperelliptic, polygonal, and reducible ones, is followed by one on non-linear series, preparatory to the use of abelian integrals and theta functions in the study of correspondences. Parts here are crowded and much too condensed, but the chapter on correspondences is more satisfactory. The next chapter frankly asserts that it is only a sketch as an introduction to the concept of rationalization. For example, complex multiplication is limited to $p=1$ and is disposed of in a dozen lines. The short chapter on rational curves is almost entirely formal; it starts from properties of the binary form, and confines the discussion of curves to a few things that fit this scheme. Two chapters on postulation and transformation of linear systems of curves are instructive and valuable. The proofs are frequently applicable to what the author calls the general case, but without pointing out wherein the restrictions lie. This is also true of the following chapter, which does not pretend to be complete.

In connection with the statement concerning the differences held (at one time) between Severi and Study on the validity of results obtained by the methods of enumerative geometry, it may be of interest to disclose that Study later wrote Severi, explicitly stating that he wished to declare that the results obtained by Severi were correct and the methods completely rigorous. I do not know that this letter was ever made public.

The last three chapters are on Cremona transformations. It is a rapid survey of the whole field, too brief to be accurate or complete. But, on the whole, it is an excellent introduction to the subject. With the exception of less than two pages, this could have been put much earlier, and would have avoided unnecessary restrictions in theorems and in the point of view. A great many references are given, usually to the sources, but occasionally to the details of a discussion of which the knowledge is assumed. At times this works a hardship; a few such are used that should not be assumed to be accessible to the reader. In a few instances, important contributions had been overlooked; for example, the essays of Field on topology, that of Nichols on jacobians, and some on existence theorems of curves with given singularities.

The typographical work is excellent and the proof reading has been well done. Most of the errors noticed would not cause confusion.

On page 272 zero and infinity are at odds; on page 300 there are a number of errors, not all purely typographical. In the references, the spelling of the name of the institution at Palermo and of other Italian names is sometimes faulty, but the citations are otherwise almost without exception correct. Throughout the book the method of referring to theorems already established is confusing in the use of capital letters. The citations made are collected with full references at the end of the volume, which is also provided with an index.

Although this review contains some frank criticisms, this book is nevertheless a valuable addition to the literature on algebraic curves.

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