SOLUTIONS OF BOUNDED VARIATION OF THE FREDHOLM-STIELTJES INTEGRAL EQUATION*

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The purpose of this note is to give conditions under which the Stieltjes integral equation

$$\phi(x) = f(x) + \lambda \int_a^b K(x, y) d\phi(y) \tag{1}$$

has a solution $\phi(x)$ of bounded variation.

In the first theorem conditions on $f(x)$ and $K(x, y)$ are given under which the method of successive substitutions yields a solution of bounded variation for a limited range of values of λ.

With further restrictions on $K(x, y)$, it is shown in the second theorem that the Fredholm method applies and the solution of bounded variation thus obtained is valid for all values of λ except for the characteristic values.

Finally, an example is given to show that the more restrictive conditions on $K(x, y)$ given in the second theorem are not sufficient to make the problem a special case of that treated by Riesz.†

THEOREM 1. *If*

(a) $f(x)$ *is of bounded variation*, $a \leqq x \leqq b$,

(b) $K(x, y)$, *defined and bounded on* $R(a \leqq x \leqq b,\ a \leqq y \leqq b)$, *is continuous in* y *for each* x *and has a total variation in* x *for each* y, $T_K(y)$, *which is a bounded function of* y *having the least upper bound* T_K *and*

(c) $$|\lambda| < 1/T_K,$$

then the function $\bar{\phi}(x)$ *defined by the series*

$$\bar{\phi}(x) = f(x) + \lambda \int_a^b K(x, y_1) df(y_1) + \lambda^2 \int_a^b K(x, y_1) d \int_a^b K(y_1, y_2) df(y_2) + \cdots \tag{2}$$

is the unique solution of bounded variation of integral equation (1).

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† F. Riesz, *Über lineare Funktionalgleichungen*, Acta Mathematica, vol. 41 (1918), pp. 71–98.

This theorem is proved by the usual method of successive substitutions. Let the function $\psi(x)$ be defined by

$$\psi(x) = \int_a^b K(x, y) df(y).$$

Designate the total variations of $f(x)$ and $\psi(x)$ by T_f and T_ψ respectively and let $M \geqq |K(x, y)|$. Then we have

$$|\psi(x)| \leqq MT_f \text{ and } T_\psi \leqq T_K \cdot T_f.$$

With the use of these inequalities it is found that series (2) obtained from equation (1) by repeated substitution converges absolutely and uniformly in x for all values of λ satisfying the inequality $|\lambda| < 1/T_K$. The function thus defined, $\bar{\phi}(x)$, is found by substitution to satisfy equation (1). That it is the unique solution follows as a consequence of the fact that the method of successive substitutions when applied to the homogeneous equation

$$\phi(x) = \lambda \int_a^b K(x, y) d\phi(y)$$

yields as its only solution $\phi(x) \equiv 0$.

THEOREM 2. *If $f(x)$ is of bounded variation in the interval (a, b) and $K(x, y)$ together with $\partial K(x, y)/\partial x$ are continuous functions of x and y in R, then there exists a unique solution $\phi(x)$ of bounded variation of equation* (1) *for all values of λ except for the characteristic values.*

This theorem can be proved by applying Fredholm's method to equation (1). However, we shall employ the following transformation* which reduces the problem to the solution of a Riemann integral equation. Let $\theta(x) = \phi(x) - f(x)$. Then equation (1) becomes

$$\theta(x) = \lambda \int_a^b K(x, y) df(y) + \lambda \int_a^b K(x, y) d\theta(y). \tag{3}$$

It is easily verified, with the given hypotheses on the functions involved, that each term of the right hand side of equation (3) is a continuous function of x and possesses a continuous

* J. D. Tamarkin suggested to me the possibility of transforming equation (1) into a Riemann integral equation.

derivative. Hence on placing $\lambda\int_a^b K(x, y)df(y) = F(y)$, we obtain from equation (3) by differentiation

$$\theta'(x) = F'(x) + \lambda \int_a^b \frac{\partial}{\partial x}K(x, y)\theta'(y)dy. \tag{4}$$

The function $F'(x)$ is continuous, and, moreover, by hypothesis $\partial K(x, y)/\partial x$ is continuous in x and y. Hence the Riemann integral equation (4) has a unique continuous solution $\theta'(x)$ for all values of λ except for the characteristic values. The unique solution of bounded variation $\phi(x)$ of equation (1) can thus be found.

An example. Let $\|f\|$ denote the maximum of the absolute value of the continuous function $f(x)$ in (a, b). One of the conditions on the transformation

$$T[f] = \int_a^b K(x, y)df(y)$$

as given by F. Riesz* is that there exists a constant M such that for all continuous functions $f(x)$

$$\|T[f]\| \leqq M\|f\|. \tag{5}$$

The function $K(x, y)$ defined in the following example satisfies the conditions of Theorem 2 whereas inequality (5) given by Riesz is not satisfied by the transformation $T[f]$. We define $K(x, y)$ to be a function of one variable, thus:

$$K(x, 0) = 0, \quad K(x, y) = y \sin(\pi/y), \ (0 < y \leqq 1).$$

We next define a sequence of continuous functions $f_n(y)$ bounded in n and y. The nth member of this sequence is a function whose graph consists of a series of broken lines. These lines have a slope equal to zero in the interval $(0, 1/(n+1))$, while in the interval $(1/(n+1), 1)$ they have a negative slope where the function $y \sin(\pi/y)$ is negative and a positive slope where this function is positive. Let δ_i denote 1 or 0 according as i is odd or even. Then

$$f_n(y) = (-1)^k k(k+1)y + (-1)^{k+1}k + \delta_k,$$
$$1/(k+1) \leqq y \leqq 1/k, \ (k = 1, 2, \cdots, n),$$

* F. Riesz, loc. cit., p. 72.

and

$$f_n(y) = \delta_n, \qquad (0 \leqq y \leqq 1/(n+1)).$$

From this definition we have

$$\frac{df_n(y)}{dy} = (-1)^k k(k+1), \qquad (1/(k+1) < y < 1/k,\ k = 1, 2, \cdots, n),$$
$$= 0, \qquad (0 < y < 1/(n+1)). \tag{6}$$

The transformation

$$T[f_n] = \int_0^1 y \sin(\pi/y) df_n(y), \quad (n = 1, 2, \cdots),$$

becomes from the definition of $f_n(y)$ and from (6)

$$\begin{aligned} T[f_n] &= \sum_{k=1}^{k=n} \int_{1/(k+1)}^{1/k} y \sin(\pi/y)(-1)^k k(k+1) dy \\ &= \sum_{k=1}^{k=n} k(k+1) \int_{1/(k+1)}^{1/k} (-1)^k y \sin(\pi/y) dy. \end{aligned} \tag{7}$$

The $\int_{1/(k+1)}^{1/k} y \sin(\pi/y)\, dy$ is in absolute value greater than the area of the triangle whose vertices are at $(1/(k+1), 0)$, $(1/k, 0)$, $[2/(2k+1), (-1)^k 2/(2k+1)]$. Since the area of this triangle is $1/[k(k+1)(2k+1)]$, we have

$$\int_{1/(k+1)}^{1/k} (-1)^k y \sin(\pi/y) dy > 1/[k(k+1)(2k+1)].$$

Consequently we get from (7)

$$T[f_n] > \sum_{k=1}^{k=n} k(k+1)/[k(k+1)(2k+1)] = \sum_{k=1}^{k=n} 1/(2k+1),$$

from which it follows that $T[f_n]$ becomes infinite with n. Hence condition (5) is not satisfied and equation (1) is not a special case of that treated by F. Riesz.

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