NEW DIOPHANTINE AUTOMORPHISMS*

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1. *Introduction.* In this note we consider the construction of an infinite set of homogeneous polynomials f such that

$$f(Y_1, \cdots, Y_n) = f^h(y_1, \cdots, y_n), \tag{1}$$

where

$$Y_i = Y_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n), \tag{2}$$

are homogeneous of degree h. In that case, f is said to admit of a diophantine automorphism.‡ The construction depends on a very simple principle connected with invariant theory.

Let ϕ be a binary form of degree δ:

$$\phi = a_x^\delta = a_0 x_1^\delta + \binom{\delta}{1} a_1 x_1^{\delta-1} x_2 + \cdots + \binom{\delta}{\delta} a_\delta x_2^\delta;$$

let q be a quadratic covariant of ϕ of degree ρ; then ψ, the discriminant of q, is an invariant of the ground-form ϕ of degree 2ρ. By the property of invariance, if under the linear transformation

$$x_1 = \alpha_{11} X_1 + \alpha_{12} X_2, \quad x_2 = \alpha_{21} X_1 + \alpha_{22} X_2, \tag{3}$$

of determinant $\Delta = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$, the form ϕ becomes

$$\Phi = A_X^\delta = A_0 X_1^\delta + \binom{\delta}{1} A_1 X_1^{\delta-1} X_2 + \cdots,$$

and ψ becomes $\Psi = \psi(A)$, then

$$\psi(A) = \Delta^{\delta\rho}\psi(a). \tag{4}$$

Now putting $q = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2$, we take in (3)

$$\alpha_{11} = q_{12}, \quad \alpha_{12} = q_{22}, \quad \alpha_{21} = -q_{11}, \quad \alpha_{22} = -q_{12}.$$

Evidently (4) becomes

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‡ E. T. Bell, this Bulletin, vol. 33 (1927), pp. 71–80.

$$\psi(A) = \psi^{\delta\rho+1}(a), \tag{5}$$

where, by virtue of the specialized transformation (3),

$$A_i = A_i(a_0, \cdots, a_\delta) \tag{6}$$

are homogeneous of degree $\delta\rho+1$.

Clearly (5) and (6) furnish a solution of (1) and (2). And (for $\psi \neq$ square) we can assert immediately that it is surely non-trivial if δ is odd. Indeed the solution is trivial only if (6) reduces to

$$A_i = \alpha\psi^{\delta/2}(a)\cdot a_i.$$

2. *Expression of q Symbolically.* It is here convenient to make use of the symbolism of the theory of invariants.* Let us set

$$\phi = a_x^\delta = (a_1x_1 + a_2x_2)^\delta, \quad q = q_x^2 = (q_1x_1 + q_2x_2)^\delta,$$

so that upon applying (3), in specialized form, we obtain

$$\begin{aligned} a_x \to A_X &= a_1(q_1q_2X_1 + q_2^2X_2) + a_2(-q_1^2X_1 - q_1q_2X_2) \\ &= (a_1q_2 - a_2q_1)(q_1X_1 + q_2X_2) = (aq)q_X, \end{aligned} \tag{7}$$

and therefore ϕ becomes

$$\Phi = (aq)(aq') \cdots (aq^{(\delta-1)})q_Xq'_X \cdots q_X^{(\delta-1)},$$

where $q, q', \cdots$ are equivalent symbols. Similarly

$$q_x \to q_1(q_1'q_2'X_1 + q_2'^2X_2) + q_2(-q_1'^2X_1 - q_1'q_2'X_2) = (qq')q_X',$$

and therefore q becomes

$$Q = (qq')(qq'')q_X'q_X'' = \psi q, \tag{8}$$

as follows immediately from the fundamental identity of invariant theory.

3. *Consequences of* (7) *and* (8). The equations (7) and (8) imply an interesting result concerning the transformation (6). Clearly all the operations performed on ϕ may be performed on Φ. By (8), the transformation (3), properly specialized, changes (7) to

$$\psi^{\delta\rho/2}\cdot(aq)(qq')q_x' = \tfrac{1}{2}\psi^{\delta\rho/2}(qq')^2a_x = \psi^{(\delta\rho+2)/2}a_x,$$

* See Grace and Young, *Algebra of Invariants*, 1903, Chap. 1.

and

$$\Phi \longrightarrow \psi^{\delta(\delta\rho+2)/2}\phi;$$

that is, Φ is transformed into ϕ multiplied by a factor free of x. This is equivalent to

$$A_i(A) = \psi^{\delta(\delta\rho+2)/2}\cdot a_i, \tag{9}$$

where the exponent is surely integral, for when δ is odd ρ must be even. It follows that the equation (6) *defines a Cremona transformation of period two.*

Now from (9)

$$\left|\frac{\partial A_i(A)}{\partial A_j}\right| \cdot \left|\frac{\partial A_i}{\partial a_j}\right| = \left|\frac{\partial(\psi^{\delta(\delta\rho+2)/2}a_i)}{\partial a_j}\right|,$$

since the determinants involved are jacobians. But the jacobian on the right is easily shown to be equal to

$$(\delta\rho + 1)^2\psi^{\delta(\delta+1)(\delta\rho+2)/2}.$$

Hence, if ψ is algebraically irreducible,

$$\left|\frac{\partial A_i}{\partial a_j}\right| = c\psi^{\delta(\delta+1)/2},$$

where c denotes a numerical constant.

4. *Special Values of* δ. We now consider some special values of δ. If δ be odd, it is evident, to begin with, that a q is always furnished by the covariant

$$K_{\delta-1} = (\phi, \phi)^{\delta-1},$$

the $(\delta-1)$th transvectant of ϕ with itself. The quantity $K_{\delta-1}$ is an irreducible covariant, and its discriminant

$$\psi = \tfrac{1}{2}(K_{\delta-1}, K_{\delta-1})^2$$

is an irreducible invariant of degree four. By what precedes, the form ψ admits of a non-trivial automorphism.

Take $\delta = 3$. Then K_2 is the Hessian of ϕ, and ψ is the discriminant of ϕ as well as of K_2. By (7) we must here consider the transformed form

$$
\begin{aligned}
(aq)(aq')(aq'')q_x q_x' q_x'' \\
&= \tfrac{1}{2}(aq)q_x((aq')^2 q_x''^2 + (aq'')^2 q_x'^2 - (q'q'')^2 a_x^2) \\
&= q\cdot(aq)(aq')^2 q_x - \psi\cdot(aq)a_x^2 q_x.
\end{aligned}
$$

But the first term involves $((\phi, q)^2, q)^1$ and $(\phi, q)^2$ vanishes, as can be seen directly or by making use of the known result that the cubic has no linear covariants. Hence the transformed form is, but for a numerical factor, the product of the discriminant into the cubicovariant. In this case, then, (6) reduces to

$$\frac{1}{\psi} A_i = f_i(a_0, \cdots, a_3),$$

where f_i is of degree three.

5. *The Case* $\delta = 5$. When $\delta = 5$, there are three quadratic covariants in the irreducible set.* The simplest is the form $K_4 = (\phi, \phi)^4$, which is generally denoted by i. For the transformed form (7) we have here $(i = i_x^2 = i_x'^2 = \cdots)$

$$
\begin{aligned}
(10)\quad (ai)(ai')\cdots(ai^{\text{iv}})i_x i_x' \cdots i_x^{\text{iv}} \\
&= \tfrac{1}{4}(ai)i_x(2(ai')^2\cdot i - Aa_x^2)(2(ai'')^2\cdot i - Aa_x^2) \\
&= (ai)(ai')^2(ai'')^2 i_x\cdot i^2 + AF,
\end{aligned}
$$

where $A = (i, i)^2$ is used in place of ψ, and F is an (integral) covariant whose exact form is immaterial. Now

$$
\begin{aligned}
(ai)(ai')^2(ai'')^2 i_x &= (((\phi, i)^2, i)^2, i)^1 \\
&= (-(j, i)^2, i)^1 = -(\alpha, i)^1 = -\beta,
\end{aligned}
$$

where j, α and β are irreducible covariants of order 3, 1, 1, respectively.† It is then clear that (10) is not a multiple of A and therefore in this case the transformation (6) is not reducible; that is, it is actually a Cremona transformation of degree $2\cdot 5+1 = 11$.

We have defined j above as $-(\phi, i)^2$; the second transvectant of j with itself is a second irreducible quadri-covariant, $\tau = (j, j)^2$. We now have, in place of (10),

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* See Grace and Young, loc. cit., p. 131.

† See Grace and Young, loc. cit.

$$(11)\quad (a\tau)\cdots(a\tau^{\mathrm{iv}})\tau_x\cdots\tau_x^{\mathrm{iv}}$$
$$= \tfrac{1}{4}(a\tau)\tau_x(2(a\tau')^2\tau - Ca_x^2)(2(a\tau'')^2\tau - Ca_x^2)$$
$$= (a\tau)(a\tau')^2(a\tau'')^2\tau_x + CF,$$

where $C=(\tau,\tau)^2$ replaces ψ. But

$$(a\tau)(a\tau')^2(a\tau'')^2\tau_x = (((\phi,\tau)^2,\tau)^2,\tau^1),$$
$$(\phi,\tau)^2 = -\,i\alpha - \tfrac{2}{3}A_j.$$

It is easy to show, using the tables of transvectants in Grace and Young, that $(i\alpha,\tau)^2$ is expressible linearly in terms of $B\alpha$ and δ (B is an invariant, α and δ are linear covariants in the irreducible set of concomitants of ϕ); $(B\alpha,\tau)^1$ and (δ,τ) are expressible in terms of $C\beta$ and $B\gamma$, γ a third linear covariant. It is then evident that (11) is not divisible by C, and therefore the transformation (6) is here also of maximum degree $(6\cdot 5+1=31)$.

The remaining quadratic covariant $\theta=(i,\tau)$ may be treated in exactly the same way, and again it appears that (6) does not reduce to a transformation of lower degree.

6. *The Case* $\delta=7$. When $\delta=7$, there are several quadratic covariants. We limit ourselves to the $K_6=(\phi,\phi)^6=q$, $\psi=(q,q)^2$. For this case, (7) becomes

$$(12)\ (aq)\cdots(aq^{\mathrm{vi}})q_x\cdots q_x^{\mathrm{vi}} = (aq)(aq')^2(aq'')^2(aq''')^2q_x\cdot q^3+\psi F.$$

Now $(\alpha q)(\alpha q')^2(\alpha q'')^2(\alpha q''')^2q_x=(\phi,q^4)^7$ and this is a member of the irreducible set of concomitants of the septimic.* Accordingly (12) is not divisible by ψ, and again (6) is of maximum degree $(2\cdot 7+1=15)$.

In general, if δ be odd $(=2k+1)$ and greater than three, then for $q=K_{\delta-1}$, (7) reduces to

$$(\phi,q^{k+1})^{2k+1}\cdot q^{k-1}+\psi F,$$

and it seems likely that this is not divisible by ψ, so that (6) would in this case always be of maximum degree $=2\delta+1$.

7. *Even Values of* δ. When δ is even it is necessary to consider covariants of somewhat higher degree. Thus for $\delta=6$ the simplest q is†

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* See F. von Gall, Mathematische Annalen, vol. 31 (1888), p. 320.

† See Grace and Young, loc. cit., p. 156.

$$q = l = (\phi, i)^4, \quad i = (\phi, \phi)^4.$$

Then for (7) we get as above

$$(al)^2(al')^2(al'')^2 \cdot l^3 + \psi F,$$

where $\psi = (l, l)^2$. But

$$(al)^2(al')^2(al'')^2 = (\phi, l^3)^6,$$

which is a member of the irreducible set of concomitants distinct from ψ.

8. *Conclusion*. The method outlined in §1 is by no means restricted to binary forms ψ. It is indeed obvious that starting with any quadratic covariant of a form ϕ in any number of variables we may arrive at equations (5) and (6). The transformation involved will certainly not be trivial if the degree of ϕ is not divisible by the number of variables.*

An obvious instance of this is furnished by quadratic forms, and indeed it is sufficiently obvious that their discriminants have the automorphic property. Another simple though less obvious instance is furnished by the quadri-covariant of the quaternary cubic ($\phi = a_x^3 = b_x^3 = \cdots$):

$$(abcd)(abce)(adef)(bdef)c_x f_x.$$

By the remark made in the last paragraph, this will certainly not lead to a trivial transformation.

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* It is of course assumed that the discriminant of the quadri-covariant of ϕ is not a kth power, k being the number of variables.