NOTE ON AN APPLICATION OF METRIC GEOMETRY TO DETERMINANTS

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1. *Introduction*. This note refers to a previous paper* published in this Bulletin. On page 754 of that paper, the following theorem is stated.

THEOREM. *If the symmetric determinant*

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12} & r_{13} & r_{14} \\ 1 & r_{21} & 0 & r_{23} & r_{24} \\ 1 & r_{31} & r_{32} & 0 & r_{34} \\ 1 & r_{41} & r_{42} & r_{43} & 0 \end{vmatrix}, \quad (r_{ij} = r_{ji}),$$

with $r_{ij}>0$, $(i, j=1, 2, 3, 4)$, $i\neq j$, *is different from zero, and the complementary minors of four of the elements in the principal diagonal vanish, then the complementary minor of the remaining element does not vanish.*

In order to prove this theorem, two cases, A and B, were considered. In Case A it was supposed that all four of the bordered complementary minors vanished. It was then shown that the fifth complementary minor (the unbordered minor that is the complementary minor of the element appearing in the first row and first column) did not vanish.

In Case B it was assumed that three of the four bordered minors and the unbordered minor, denoted by $\mathcal{E}(p_1, p_2, p_3, p_4)$, vanished. It was stated that Case B was exhausted by a study of two sub-cases, both of which were examined and found to contradict the hypotheses stated for Case B. It was concluded, then, that this case could not exist, and from this conclusion four interesting corollaries were stated. The principal interest of the paper seems to the writer to lie in these corollaries.

A communication received from W. V. Parker called my attention to the fact that the hypotheses explicitly made upon

* This Bulletin, vol. 37 (1931), pp. 752–758.

the elements of the determinant D, while sufficient for the validity of the theorem, are not sufficient to prevent the possibility of Case B occurring, and hence, the corollaries, dependent upon the impossibility of Case B for their correctness, are not true without an additional hypothesis upon D. This Parker showed by exhibiting a determinant which actually satisfied all the requirements stated, and which came under Case B.

In the present paper we give a brief proof of the theorem with the conditions as stated in the original paper, in order to clear up any doubts that might exist about the validity of the theorem itself. Further, we make explicit the additional hypothesis upon the elements of the determinant D (implicit in the previous paper) and show in some detail that with this hypothesis the two cases examined under Case B are indeed the only two types that cannot be immediately rejected. These having been proved impossible, the corollaries are seen to be valid. Finally, the last part of this note is concerned with obtaining the only class of determinants, of which the one due to Parker is a special case, for which the Case B *does* exist. It remains for me to express my thanks to him for his interest and his contribution.

2. *Proof of the Theorem.* To prove the theorem itself, we consider again the two cases A and B. The elements r_{ij} having been replaced by $(ij)^2$, where (ij) is a positive number which may be thought of as representing the distance between a point p_i and a point p_j of a quadruple p_1, p_2, p_3, p_4, the assumption in Case A that all four bordered complementary minors vanish means, as stated in the previous paper, that all four of the triples of points contained in the four points p_1, p_2, p_3, p_4 are linear. The determinant D being, by hypothesis, different from zero, the four points are not linear, and hence they form a pseudo-linear quadruple. (This is a result due to Karl Menger, explicit reference to which was given.) The determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$ formed for a pseudo-linear quadruple is shown to be different from zero, and the theorem is proved for this case.

In Case B we suppose that the four complementary minors assumed to vanish consist of three of the bordered minors and the unbordered minor $\mathcal{E}(p_1, p_2, p_3, p_4)$. We must then show that the remaining minor, a bordered one, does not vanish. This is immediate, for suppose that this minor vanishes. Then all four of the bordered minors vanish, and hence the four points p_1, p_2, p_3, p_4 are either linear or pseudo-linear. But, since the determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$ is assumed to vanish, the points are not pseudo-linear, for, from Case A we have seen that the determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$ formed for a pseudo-linear quadruple does not vanish. Therefore, the four points must be linear. But this is impossible, since the determinant D vanishes for a linear set of points, and, by hypothesis, the determinant D does not vanish. Hence, the assumption that the remaining minor vanishes is seen to lead to a contradiction, and the theorem is proved.

3. *The Triangular Inequality.* We now state the hypothesis that is sufficient to prevent the existence of Case B. This hypothesis is that *the elements $r_{ij}=(ij)^2$ of D are such that each of the four triples p_1, p_2, p_3; p_1, p_2, p_4; p_1, p_3, p_4; p_2, p_3, p_4 contained in the four points p_1, p_2, p_3, p_4 satisfy the triangular inequality.** This geometric way of stating the condition is made use of because of its brevity; the condition may, of course, be stated in algebraic language.

It is now to be shown that with this condition adjoined to those made explicitly in the theorem, the Case B cannot exist. Since the theorem itself has been established, we know that in the Case B, the remaining complementary minor does not vanish. This is a bordered minor, and its non-vanishing means that the triple for which it is formed is not linear. We may assume the labelling so that this non-linear triple is p_1, p_2, p_4. The other three triples being linear (since their minors vanish) we have

$$(123)(231)(312) = 0, \quad (234)(342)(423) = 0, \quad (341)(413)(134) = 0,$$

where by the symbol ijk we mean $ij+jk-ki$, and the parentheses indicate multiplication.

There are, then, 27 possible cases to examine, corresponding to the different ways in which the three bordered minors can vanish. An examination of these 27 cases yields the following results:

Twelve of the cases result in the fourth triple p_1, p_2, p_4 being linear, and hence are to be rejected, since we have seen that this

* Three points are said to satisfy the triangular inequality if their distances are such that the sum of any two is greater than or equal to the third.

triple is not linear, since the determinant formed for it does not vanish.

Nine of the cases result in the fourth triple p_1, p_2, p_4 not satisfying the triangular inequality, and are therefore to be rejected since we have supposed that all of the triples satisfy this inequality.

Two of the combinations are to be rejected since they imply that the sum of three distances equals zero.

This leaves four cases to be examined. These four cases are:

$$\text{(a)} \begin{cases} 132 = 0, \\ 134 = 0, \\ 234 = 0, \end{cases} \quad \text{(b)} \begin{cases} 123 = 0, \\ 234 = 0, \\ 341 = 0, \end{cases} \quad \text{(c)} \begin{cases} 312 = 0, \\ 342 = 0, \\ 134 = 0, \end{cases} \quad \text{(d)} \begin{cases} 231 = 0, \\ 423 = 0, \\ 413 = 0. \end{cases}$$

Case (a) is seen to reduce to sub-case alpha of Case B if we interchange the labels of p_3 and p_1. Hence, it has been shown to be impossible. Case (b) is precisely the case treated in sub-case beta of Case B, and there shown to be impossible. Case (c) is merely another labelling of case (b) in which p_1 and p_2 are interchanged. Case (d) is reduced to case (c) if the letters p_2 and p_4 are interchanged. Hence these four cases are also impossible and the Case B cannot exist. Therefore, with the additional assumption stated in this section, the corollaries of the theorem are valid.

4. *Possibility of Case* B. We now obtain a class of determinants that may occur under Case B when we drop the assumption that each of the triples satisfies the triangular inequality.

Consider the relations $312=0$; $234=0$; $413=0$. If we put $12=a$, $14=c$, $24=b$, we obtain, by substitution in the above three equations, $13=(1/2)(-a+b-c)$, $23=(1/2)(a+b-c)$, $34=(1/2)(-a+b+c)$. Since the distance (ij) is always positive, we have the positive numbers a, b, c satisfying the inequalities $a+b>c$, $b+c>a$, $b>a+c$. It will be observed that the triple p_1, p_2, p_4 does not satisfy the triangular inequality. Now, by hypothesis, the determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$ vanishes. Substituting the values obtained above, and developing the determinant, we obtain

$$\mathcal{E}(p_1, p_2, p_3, p_4) = (1/16)(a+b+c)(a+b-c)$$
$$\cdot(a-b+c)(-a+b+c)^3(2ab+2ac+2bc-a^2-b^2-c^2).$$

Thus, it is seen, $\mathcal{E}(p_1, p_2, p_3, p_4)$ is zero if and only if

$$a^2 + b^2 + c^2 - 2(ab + ac + bc) = 0,$$

since the first four parentheses are seen not to vanish by virtue of the inequalities that a, b, c satisfy.

Let us calculate the value of the determinant D formed for this case. It is found that

$$D(p_1, p_2, p_3, p_4) = -\tfrac{1}{8}(-a^3 - 2b^3 - 2c^3 + 3ab^2 + 3ac^2 + 2bc^2 + 2b^2c - 6abc)^2,$$

which may be written in the form

$$D = -\tfrac{1}{8}[-(a^2+b^2+c^2-2ab-2ac-2bc)(a+2b+2c)-16abc]^2,$$

and, on account of the condition obtained by setting $\mathcal{E}$ equal to zero, this becomes $D = -32a^2b^2c^2$.

In order to see that this constitutes the only class of determinants for which Case B exists, it is necessary and sufficient to examine each of the nine combinations referred to in §3 for which the fourth triple p_1, p_2, p_4 does not satisfy the triangular inequality. These nine combinations are

$$(1)\ \begin{cases}312 = 0,\\ 234 = 0,\\ 413 = 0,\end{cases} \quad (2)\ \begin{cases}123 = 0,\\ 423 = 0,\\ 134 = 0,\end{cases} \quad (3)\ \begin{cases}231 = 0,\\ 342 = 0,\\ 341 = 0,\end{cases}$$

$$(4)\ \begin{cases}123 = 0,\\ 342 = 0,\\ 134 = 0,\end{cases} \quad (5)\ \begin{cases}123 = 0,\\ 234 = 0,\\ 413 = 0,\end{cases} \quad (6)\ \begin{cases}312 = 0,\\ 423 = 0,\\ 134 = 0,\end{cases}$$

$$(7)\ \begin{cases}312 = 0,\\ 234 = 0,\\ 341 = 0,\end{cases} \quad (8)\ \begin{cases}231 = 0,\\ 342 = 0,\\ 413 = 0,\end{cases} \quad (9)\ \begin{cases}231 = 0,\\ 423 = 0,\\ 341 = 0.\end{cases}$$

The first combination is the one just treated, and shown to exist. It is seen at once that the second combination may be obtained from the first by interchanging the letters p_1 and p_2; while the third is similarly obtained from the first by interchanging the labels of p_1 and p_4. Hence these two combinations are equivalent to the first and do not require further treatment.

It will be found upon examination that the last six combinations are equivalent to the fourth combination, being obtainable from this combination by suitable changes in the labelling of the points. It will be sufficient, then, to examine combination (4).

We have, writing $12=a$, $14=c$, $24=b$, and substituting in the relations (4), that

$$13 = \tfrac{1}{2}(a + b + c), \quad 23 = \tfrac{1}{2}(- a + b + c),$$
$$34 = \tfrac{1}{2}(- a - b + c),$$

the positive numbers a, b, c satisfying the inequality $c>a+b$. The determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$, which vanishes, by hypothesis, takes the form, upon being developed,

$$\begin{aligned}\mathcal{E}(p_1, p_2, p_3, p_4) &\\ = (1/16)&(a + b + c)^2(- a + b + c)^2(a + b - c)^2 \\ &\cdot(a^2 + b^2 + c^2 + 2ab + 2bc - 2ac).\end{aligned}$$

Now evidently none of the parentheses vanishes, and hence $\mathcal{E}(p_1, p_2, p_3, p_4)$ is not zero, which contradicts the hypothesis. Hence, this case is not possible.

Thus we have seen that if we assume the elements $r_{ij}=(ij)^2$ of the determinant D to be such that each triple of points satisfies the triangular inequality, then only Case A is possible, and the four corollaries, stated at the end of the paper about which this note is written, are valid. If we do not make this assumption, then Case B can exist for only one type of combination, the one investigated in this paper. We notice that in either event, the value of the determinant is given by $-32a^2b^2c^2$, where a, b, c are positive numbers satisfying the appropriate conditions.

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