ON CANONICAL BINARY TRILINEAR FORMS*

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1. *Introduction.* In a paper which appeared in 1922 Fraülein E. Schwartz,† using transvectants as invariants, obtained the classes of all binary trilinear forms equivalent in the complex field, and exhibited a representative canonical form for each class. A few papers‡ have since appeared on the algebraic invariants of these forms. The purpose of this note is to show that these forms can be obtained rather directly from the theory of pairs of bilinear forms without the use of algebraic invariants. The method gives rise to two arithmetic invariant properties of the 3-way matrices formed by the coefficients of these forms, one of which is the precise generalization of ordinary matrix rank.§ No claim is made to originality of results; the paper is rather an immediate application of the known theory on pairs of bilinear forms.

2. *Definitions.* Let $T=a_{ijk}x_iy_jz_k$ be a binary trilinear form with which is associated the cube matrix $\alpha=(a_{ijk})$, $(i,j,k=1,2)$. The ρ-files of α are defined to be the one-way arrays of elements obtained by fixing all but the index $\rho(\rho=i,j,k)$ of α. The rank r_ρ of α on the index ρ is the number of linearly independent ρ-files of α. Evidently file ranks are invariant under non-singular

* Presented to the Society, April 9, 1932.

† E. Schwartz, *Über binäre trilineare Formen*, Mathematische Zeitschrift, vol. 12 (1922), pp. 18–35. Among previous papers on these forms we find:

LePaige, *Sur la théorie des formes binaires à plusieurs séries de variables*, Bulletin de l'Académie Royale, (3), vol. 2 (1881), pp. 40–53.

R. Dedekind, *Über binäre trilineare Formen und die Komposition der binären quadratischen Formen*, Journal für Mathematik, vol. 129 (1905), pp. 1–34.

‡ W. Saddler, *Triple binary forms*, Proceedings of the Cambridge Mathematical Society, vol. 22 (1925), pp. 688–693.

C. W. Gilham, *The complete system of concomitants of a multiple binary form*, Journal of the London Mathematical Society, vol. 4 (1929), pp. 166–471.

For a geometric treatment of these forms see A. Duschek, *Eine Abbildung der binären trilinear Form*, Jahresbericht der Vereinigung, vol. 32 (1924), pp. 234–239.

§ F. L. Hitchcock, *Multiple invariants and generalized rank of a p-way matrix or tensor*, Journal of Mathematics and Physics, vol. 6 (1927), No. 3, p. 171.

linear transformations. The rank of α and T is the invariant rank set (r_i, r_j, r_k)

A couche of α on the index ρ is defined to be a 2-way minor of α obtained by assigning a fixed value to the index ρ of α. If θ and Δ are ρ-couches, where Δ is non-singular, the determinant $|\theta-\lambda\Delta|$ is called the ρ-invariant factor of α and T.

3. *Reductions*. Let $T=T_1z_1+T_2z_2$, where T_1 and T_2 are bilinear forms in x and y. Let the matrices of T_1 and T_2 be denoted by A and B respectively. We consider three cases.

CASE a. α of rank (2, 2, 2). T non-decomposable.

Suppose that A is non-singular. T_1 and T_2 can be reduced under non-singular linear transformations* on x and y to one of the two pairs of forms

$$\text{(a.)}\qquad x_1'y_1' + x_2'y_2'\,;\; rx_1'y_1' + sx_2'y_2',$$

$$\text{(b.)}\qquad x_1'y_1' + x_2'y_2'\,;\; rx_1'y_1' + rx_2'y_2' + x_2'y_1'.$$

Applying the transformation $t_1=z_1+rz_2$, $t_2=z_1+sz_2$ to the form (a.) we obtain

$$R = x_1'y_1't_1 + x_2'y_2't_2.$$

The determinant of the transformation is not zero since $r\neq s$. Applying the transformation $z_1'=z_1+rz_2$, $z_2'=z_2$ of determinant unity to the form (b.), we obtain the form

$$L = x_1'y_1'z_1' + x_2'y_2'z_1' + x_2'y_1'z_2'.$$

If all couches of α are singular we assume that the non-couche

$$C = \begin{pmatrix} a_{111} & a_{122} \\ a_{211} & a_{222} \end{pmatrix}$$

is non-singular. Applying the transformation $x_1'=a_{111}x_1+a_{211}x_2$, $x_2'=a_{122}x_1+a_{222}x_2$ to T we again obtain the form R.

To distinguish between the forms R and L we note that the k-invariant factor of (b.) satisfies the following property:

(c.) *The invariant factor is a perfect square*,

while the k-invariant factor of (a.) is not.

By two theorems of Dickson† property (c.) and its converse

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* L. E. Dickson, *Modern Algebraic Theories*, p. 116, Theorem 5.

† Invariance of (c.) and its converse under transformation on x and y follows from Theorem 1, p. 113, Dickson, op. cit. By the Lemma, p. 114, of the same book powers contained in k-invariant factors are transformed into powers of the same degree under transformation on z.

are invariant under transformations on x, y and z provided the invariant factor in question is defined for the transformed matrix.

It is easily shown that if a non-singular transformation is made on L, the couches on an index of the transformed matrix Ψ are not both singular. Since all of the invariant factors of L satisfy (c.), Ψ possesses on each index invariant factors which satisfy (c.). Hence (c.) suffices to distinguish between the transforms of L and R.

Since R and certain forms equivalent to R do not possess invariant factors in the sense defined, it is convenient to employ the definition of invariant factors for singular couches. If U, W are ρ-couches of α and $|\sigma U+\tau W| \not\equiv 0$, $|\sigma U+\tau W|$ is called the ρ-invariant factor of α and T. Evidently $|\sigma U+\tau W| \equiv 0$ if α is of rank unity on the index of the rows of U and W. On the other hand $|\sigma U+\tau W| \not\equiv 0$ if α is of rank 2 on all indices.

The invariant factors of forms equivalent to R satisfy the converse of (c).

Case b. α of rank (1, 2, 2). T partly decomposable.

The dependence of the i-files gives the form $T' = Sx_1'$, where S is a bilinear form in y and z with matrix D. Transforming on z with the matrix D^{-1}, we get

$$H = x_1' y_1' z_1' + x_1' y_2' z_2' .$$

The i-invariant factor of H satisfies the property (c.).

Case c. α of rank (1, 1, 1). T completely decomposable.

The linear dependence of files on all indices gives readily

$$K = x_1' y_1' z_1' .$$

If a form T of rank (1, 1, 2) exists, T is equivalent under transformations on x and y to the form $\phi = Px_1' y_1'$, where P is linear in z. Since the matrix of P is a vector of 2 elements, ϕ is of rank (1, 1, 1), contrary to hypothesis. We have therefore treated all cases.

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