ON PROPOSITION *4·78 OF PRINCIPIA MATHEMATICA†

BY B. A. BERNSTEIN

1. *Introduction.* Whitehead and Russell's *Principia Mathematica* contains the following proposition, derived from their theory of deduction for "elementary" propositions:

$$*4.78. \qquad \vdash\cdot \{p \supset (q \vee r)\} \equiv \{(p \supset q) \vee (p \supset r)\}\,.\ddagger$$

The authors say of this proposition that it has not "its analogue for classes." They mean by this that *4·78 is not a proposition in the logic of classes. Their reason for holding this view is this: "put $p =$ English people, $q =$ men, $r =$ women; then p is contained in 'q or r,' but is not contained in q and is not contained in r." As a proposition in the logic of propositions, the authors would interpret *4·78 as: the proposition "p implies 'q or r'" is equivalent to the proposition " 'p implies q' or 'p implies r'." It is my object (1) to point out that the authors' concrete interpretations of *4·78, both for classes and for propositions, are inadmissible, (2) to prove, by actually deriving *4·78 from the logic of classes, that the authors are mistaken in their view that *4·78 does not hold for classes, and (3) to prove that the fact of logic given by the above interpretation of *4·78 for propositions cannot be derived from the *Principia's* theory of deduction. The last fact will exhibit the serious inadequacy of the theory of deduction as "the calculus of propositions."

2. *Inadmissibility of the Principia's Interpretations of* *4·78. That the above interpretations of *4·78 for classes and for propositions are both inadmissible, can be seen easily when we consider the *Principia's* definitions of the symbols $p \supset q$ and $p \equiv q$, involved in *4·78. These definitions are:

$$*1.01. \qquad (p \supset q) = (\sim p \vee q)\ Df,$$

$$*4.01. \qquad (p \equiv q) = (p \supset q)(q \supset p)\ Df,$$

† Presented to the Society, October 31, 1931.

‡ The numbering and the notation are those of the *Principia*, except that ordinary parentheses are used for dot-parentheses.

where the symbol pq, involved in $*4\cdot 01$, is defined by

$$*3.01. \qquad pq = \sim(\sim p \vee \sim q)\ Df.$$

By the primitive propositions $*1\cdot 7$ and $*1\cdot 71$ of the theory of deduction, the symbols $\sim p$ and $p \vee q$, and hence the right-hand members of $*1\cdot 01$ and $*4\cdot 01$ are the same kinds of concepts as are p and q. If, then, p and q are classes, $p \supset q$ and $p \equiv q$ are *classes*, and hence *cannot* be interpreted, as the *Principia* interprets them, as the *propositions* "p is contained in q" and "p is equivalent to q." If p and q are elementary propositions, then $p \supset q$ and $p \equiv q$ are *elementary propositions*, and hence *cannot* be interpreted, as the *Principia* interprets them, as the *non-elementary* propositions "p implies q" and "p is equivalent to q." The *Principia*'s interpretations of $*4\cdot 78$ for classes and for propositions are, then, both inadmissible.†

3. *Derivation of* $*4\cdot 78$ *from the Logic of Classes.* In order to see that proposition $*4\cdot 78$ holds for classes, observe, first, that in Boolean form $*4\cdot 78$ is:

$$4.78.\ \{[p' + (q + r)]' + [(p' + q) + (p' + r)]\}\{[(p' + q) + (p' + r)]' + [p' + (q + r)]\} = 1.\ddagger$$

That this proposition can be derived from the logic of classes I shall show with the help of the following well known propositions in that logic:

(i) $$a + (b + c) = (a + b) + c,$$

(ii) $$a + b = b + a,$$

(iii) $$a + a = a,$$

(iv) $$a' + a = 1,$$

(v) $$a1 = a.\S$$

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† The *Principia*'s interpretation of $*4\cdot 78$ for classes is inadmissible for the further reason that it fails to give a meaning for classes of the symbol "$\vdash \cdot p$."

‡ This is obtained from $*4\cdot 78$ by using definitions $*1\cdot 01$ and $*4\cdot 01$, and by writing $p=1$ for $\vdash \cdot p$, p' for $\sim p$, $p+q$ for $p \vee q$. See my article *Whitehead and Russell's theory of deduction as a mathematical science*, this Bulletin, vol. 37, pp. 480–488.

§ See E. V. Huntington, *Sets of independent postulates for the algebra of logic*, Transactions of this Society, vol. 5 (1904), pp. 288–309. Propositions (i), (ii), (ii), (v) are respectively Huntington's XIIIa, IIIa, VIIIa, IIb; proposition (iv) is part of Huntington's V modified by IIIa.

In order to derive $4\cdot 78$ from (i)–(v), I shall first establish the following lemma.

(vi) $$(p' + q) + (p' + r) = p' + (q + r).$$

This lemma follows from (i)–(v), because

$$\begin{aligned}(p'+q) + (p' + r) &= [(p' + q) + p'] + r = [p' + (p' + q)] + r\\ &= [(p' + p') + q] + r = (p' + q) + r\\ &= p' + (q + r),\end{aligned}$$

by (i), (ii), (i), (iii), (i). Proposition $4\cdot 78$ then follows, because

$$\begin{aligned}&\{[p' + (q + r)]' + [(p' + q) + (p' + r)]\}\{[(p' + q) + (p' + r)]'\\ &\quad + [p' + (q + r)]\} = \{[p' + (q + r)]' + [p' + (q + r)]\}\\ &\quad \cdot \{[p' + (q + r)]' + [p' + (q + r)]\} = 1\cdot 1 = 1,\end{aligned}$$

by (vi), (iv), (v).

4. *The Proposition for which* $*4\cdot 78$ *Was Intended not Derivable from the Theory of Deduction.* To prove that the proposition for which $*4\cdot 78$ was intended cannot be derived from the *Principia*'s theory of deduction, observe, first, that the proposition "p implies q" is properly symbolized in Boolean language by "$p'+q=1$." For, "p implies q" means "If p is true then q is true," or, in symbols, "If $p=1$, then $q=1$," which is easily verified to be equivalent to "$p'+q=1$." If we denote "p implies q" by "$p<q$," we may then have the definition

(vii) $$(p < q) = (p' + q = 1)\ Df.$$

The proposition for which the authors of the *Principia* have intended their proposition $*4\cdot 78$ is, then, the proposition

(viii) *If* $p<q+r$, *then* $p<q$ or $p<r$, *and conversely.*

That this proposition is not derivable from the *Principia*'s theory of deduction is seen from the following independence-system for (viii) with respect to the postulates, in Boolean form, underlying the theory of deduction:†

$p, q, r, \cdots$ = the totality of closed regions in a plane region U, including U and including the "null" region Z;

p' = the region in U outside p;

$p+q$ = the smallest region which includes p and q.

† For a list of the postulates, in Boolean form, of the theory of deduction see my paper cited above.

5. *Bearing on the Nature of the Theory of Deduction.* The fact that proposition (viii) cannot be derived from the theory of deduction has an important bearing on the nature of that theory. The theory of deduction has been designed as "the calculus of propositions." Proposition (viii) is a well known proposition in the classic logic of propositions; the theory cannot yield this proposition; and so the theory cannot serve as "the calculus of propositions."

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A SUFFICIENT CONDITION FOR THE EXISTENCE OF A DOUBLE LIMIT

BY J. A. CLARKSON

In the elementary theory of limits it is often emphasized that the existence of a unique limit for a single-valued function $f(x, y)$ as the point $P(x, y)$ approaches $Q(a, b)$ along every straight line through Q does not imply the existence of the double limit

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y). \tag{1}$$

As early as 1873 Thomae* gave an example to illustrate this fact.

The question then naturally arises: Is the existence of a unique limit as P approaches Q along some more extensive class of curves sufficient to insure the existence of (1)? This question is immediately answered by the following theorem.

THEOREM. *If $f(x, y)$ has a unique limit L as $P(x, y)$ approaches $Q(a, b)$ along every curve having a tangent at Q, the double limit* (1) *exists.*

PROOF. Suppose, if possible, that it does not. Then there exists an $\epsilon > 0$ such that in any circle about Q there are points p for which

$$|f(p) - L| > \epsilon. \tag{2}$$

We denote by E the set of all such points.

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* J. Thomae, *Abriss einer Theorie der complexen Functionen*, 2d ed., Halle, 1873, p. 15.