AN EXAMINATION OF SOME CUT SETS OF SPACE*

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PART I

The purpose of this paper is to examine some pairs of points which are cut sets of a locally connected, locally compact, separable, and connected metric space S which has no single cut point. Under such an hypothesis the following statement will be proved.

If L is the set of all points (x) such that x together with some point y_x separates two fixed points a and b of the space S, then $L+a+b$ is closed and compact.†

By the pair (x, y) separating a and b is meant that there exists at least one separation $S_a+S_b=S-x-y$ such that no point of S_a is a point or limit point of S_b and no point of S_b is a limit point of S_a, where $a \subset S_a$ and $b \subset S_b$.

Two properties of S used in the proof are the following:

I. Between a and b there exists at least one pair of arcs T_x and T_y having just their end points a and b in common.‡

II. If X is any closed set, every component of $S-X$ is an arcwise connected open set with at least one limit point in X.§

Properties of simple arcs which are used are the following:

III. If x is any point of an arc ab, then ab may be written as the sum of two arcs ax and xb having just x in common.

IV. The points of an arc ab may be ordered. If it is assumed that a precedes b, $a \propto b$, the ordering gives the following relations:

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† This result is analogous to the theorem of G. T. Whyburn, this Bulletin, vol. 33 (1927), p. 685, to the effect that if, in any locally connected and metric continuum S, K is the set of all points separating two fixed points a and b, then $K+a+b$ is closed and compact. See also R. L. Wilder, this Bulletin, vol. 34 (1928), p. 649.

‡ See G. T. Whyburn, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38; and W. L. Ayres, American Journal of Mathematics, vol. 51 (1929), pp. 577–594. For a short proof of this theorem see G. T. Whyburn, this Bulletin, vol. 37 (1931), p. 429.

§ R. L. Moore, Mathematische Zeitschrift, vol. 15 (1922).

A point x precedes a point y, $x \propto y$, if and only if $x \subset ay$ and $y \subset xb$, where ab is written first as the sum of the arcs ay and yb and again as the sum $ax+xb$; if $x \propto y$, then y does not precede x; if $x \propto y \propto z$, then $x \propto z$.

V. If K is any closed set and ab any arc, the product $K \cdot ab$ has a last point on ab.

VI. If $\sum_1^\infty x_i$ is any monotonic plus* set of points on an arc ab with limit point p and z is any point of the subarc $ap=az+zp$ of ab, then zp contains all but a finite number of the points $\sum_1^\infty x_i$.

The next lemma is of importance in fixing the pairs (x, y).

LEMMA. *If T_x and T_y are any two arcs from a to b having just their end points a and b in common and (x, y) is any pair of points separating a and b, then x is contained in one arc and y in the other.*

PROOF. The assumption that one of the points is not contained in one of the arcs and the other point contained in the remaining arc easily leads to a contradiction, for then one of the arcs, say T_x, would contain neither x nor y. Thus, $T_x \subset S-x-y$, which is impossible since the pair (x, y) separates a and b while T_x is a connected set containing both a and b.†

Since a simple arc is a compact set of points, the proof that L is compact results immediately from the fact that $L \subset T_x+T_y$. Also, by choosing the order on T_x and T_y such that $a \propto b$ on both, a partial ordering of $L+a+b$ is established, e.g., a subset Q of points x of L is said to be monotonic if it is monotonic with respect to the order of T_x. As the point y also belongs to L the arcs T_x and T_y form a division of L into two parts $H_x=T_x \cdot L$ and $H_y=T_y \cdot L$. For the proof that $L+a+b$ is closed it will be assumed that a limit point p of L does not belong to $L+a+b$ and shown that this leads to a contradiction. Without loss it may be supposed that p is a limit point of a monotonic plus set of points $\sum_1^\infty x_i$ of H_x. Two main cases then arise.

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* The collection $\sum_1^\infty x_i$ is said to be *monotonic plus* if $x_i \propto x_{i+1}$ for each i. The collection is said to be *monotonic minus* provided $x_{i+1} \propto x_i$ for each i.

† From now on it will be assumed that one pair of the arcs T_x and T_y has been fixed and that the points (x, y) have been so named that $x \subset T_x$ and $y \subset T_y$.

CASE I. *The corresponding set* $\sum_1^\infty y_i$ *of points* y_i, *which together with* x_i *separate* a *and* b, *consists of a finite number of distinct points.*

If this be true, then an infinite number of the points $\sum_1^\infty x_i$ must be paired with one of the points of $\sum_1^\infty y_i$. Suppose that the pairs (x_{n_i}, y_k), where $i=1, 2, 3, \cdots$, separate a and b and the x_{n_i}'s are so labeled as to be monotonic. Now the pair (p, y_k) does not separate a and b for p is not a point of $L+a+b$. Hence, if C is the component of $S-p-y_k$ containing a, then C contains b. But, from Property II, a simple arc T, contained in C, exists from a to b. Writing $T_x=ap+pb$ and using Property V, we see that the arc T has a last point u on ap. Since $u\neq p$ the subarc up of ap contains all but a finite number of the points of $\sum_{i=1}^\infty x_{n_i}$, Property VI. Thus, some i exists such that $x_{n_i}\subset up-u$. However, this is impossible for then T would be a connected set containing both a and b and lying within $S-x_{n_i}-y_k$.

Since Case I leads to a contradiction there is left Case II.

CASE II. $\sum_1^\infty y_i$ *consists of an infinite number of distinct points.*

By choosing the x_i's so that the corresponding y_i's are monotonic on T_y, Case II may be divided into four parts:

A. *The* y_i*'s are monotonic plus with limit point* $q\neq b$.

B. *The* y_i*'s are monotonic minus with limit point* $q\neq a$.

C. *The* y_i*'s are monotonic plus with limit point* $q=b$.

D. *The* y_i*'s are monotonic minus with limit point* $q=a$.

CASE II A. Exactly as before, the component C of $S-p-q$ containing b contains a since p is not a point of $L+a+b$. Also, an arc T from a to b exists such that $T\subset C$. Writing $T_x=ap+pb$ and $T_y=aq+qb$, then, just as in Case I, we see that the arc T has a last point u on ap and a last point v on aq. Likewise, from Property VI, the subarc up contains all but a finite number of the points $\sum_1^\infty x_i$ and the subarc vq contains all but a finite number of the points $\sum_1^\infty y_i$. That is to say, there exists a number K such that $T\subset S-x_i-y_i$ if $i>K$. But this is impossible since T is a connected set containing both a and b.

CASE II B. With exactly similar reasoning to that of Case II A it may be shown that this case again leads to a contradiction.

There remain Cases II C and D, the latter of which will be treated next.*

CASE II D. From the fact that an arc minus its end point is a connected set it follows that $ax_1 - x_1 \subset S_{a_i}$ for every i, where $T_x = ax_1 + x_1 b$ and $S - x_i - y_i = S_{a_i} + S_{b_i}$, a separation of $S - x_i - y_i$ containing a and b respectively. Thus, if z is a point of $ax_1 - x_1 - a$ the pairs (x_i, y_i) separate z and b as well as a and b. Also, as $z \neq a$, the results of Case II B may be applied to the effect that the pair (a, p) separates z and b. (See also the footnote below.) It will be shown that Case II D contradicts this result.

Clearly the pair (a, p) separates x_1 and b as well as z and b. However, since p is not a point of $L+a+b$, the component C of $S - p - y_1$ containing a must contain b. But as the subarc ap of T_x minus its end point p is a connected set lying in $S - p - y_1$, it follows that the point x_1 belongs to C. Thus a simple arc T, contained in C, exists from x_1 to b. Obviously T does not contain a, for then the subarc of T from a to b would lie in $S - x_1 - y_1$. Hence $T \subset S - a - p$, which is impossible since the pair (a, p) separates x_1 and b. We have left then Case II C.

CASE II C. For this case consider a compact region V around p such that the closure $\overline{V}$ of V is contained in $S - T_y$. Just as in Case II D the component C_i of $S - p - y_i$ containing a contains both x_i and b, for p is not a point of $L+a+b$. Thus, for every i an arc T_i exists from x_i to b and lies within $S - p - y_i$. As V contains all the x_i's but a finite number let it be assumed that the x_i's used from now on are so chosen that $x_i \subset V$. Using the property that the boundary of V, $F(V)$, is closed, we see that there exists a first point q_i of T_i, in the direction from x_i to b such that $q_i \subset F(V)$. Thus, the subarc $N_i = x_i q_i$ of T_i lies entirely within V except for its end point q_i on $F(V)$.

DEFINITION. The *limit superior* N of a collection of sets (N_i) is the set of all points x, such that if R is any region containing x, R contains points from an infinite number of the sets N_i. The *limit inferior* M of the collection (N_i) is the set of all points y, such that if U is any region containing y, then U contains points from all but a finite number of the sets N_i. The collection (N_i) is said to be *convergent* and have limit $K = N$ if $N = M$.

* The results of Cases II A and B could also be stated: *If* $\sum_1^\infty x_i$ *and* $\sum_1^\infty y_i$ *are each monotonic with limit points* p *and* q, *respectively, where* $(p+q)\cdot(a+b)=0$, *then the pair* (p, q) *separates* a *and* b.

From the fact that V is compact and N_i is a continuum, the theorems on infinite collections of sets may be used to choose a sub-collection (N_{v_i}) of (N_i) which is convergent, whose limit N is a continuum, and such that the points x_{v_i} are monotonic on T_x.

The only point which N has in common with T_x is p, as is seen in the following manner. If $N \cdot ap$ contained points other than p, let such a point be z. Writing $ap = az + zp$ and using Property VI we see that zp contains all but a finite number of the points x_{v_i}. Also, if $j > k$, N_j does not contain x_k, for if it did we could write $ap = ax_j + x_j p$ and then the arcs ax_j and T_j would contain an arc from a to b which would contain neither x_j nor y_j. Hence, for n so large that $x_{v_n} \subset zp - z$ and $S_{a_{v_n}} + S_{b_{v_n}}$, a separation of $S - x_{v_n} - y_{v_n}$, the point z lies in $S_{a_{v_n}}$ while $\sum_{j=n+1}^{j=\infty} N_{v_j} \subset S_{b_{v_n}}$. But this is impossible since z is a limit point of this latter sum.

The assumption that $N \cdot pb$ contains points other than p, where pb is the remaining subarc of T_x, leads to a contradiction in a similar manner. Supposing that $z \subset N \cdot (pb - p)$, it is clear that every pair (x_i, y_i) separates a and z as well as a and b. From the note to Case IIB the pair (p, b) also separates a and z. If $S_a + S_z$ be a separation of $S - p - b$ containing a and z respectively, every one of the sets $(N_{v_i} - b)$ is contained in S_a, for $N_i - b$ is connected and $\sum_1^\infty x_{v_i} \subset S_a$. But this is impossible since a limit point of $\sum_1^\infty N_{v_i}$ is contained in S_z. Thus $N \cdot T_x = p$.

As $\overline{V}$ is compact and $F(V)$ is closed, the points q_{v_i} have a limit point q contained in $F(V)$. Thus, since $q \subset F(V)$, $q \neq p$, that is, q is not a point of T_x or T_y. Let U be a connected region containing q such that $\overline{U} \subset S - T_x - T_y$. As q is a limit point of $\sum_1^\infty q_{v_i}$, some m exists such that $q_{v_m} \subset U$. Since the arc N_{v_m} does not contain p it has a last point w on ap. By Property VI the subarc wp of ap contains all but a finite number of the points x_i. Choose x_{v_n} such that $x_{v_n} \subset wp - w$ and $n > m$. Since the x_{v_i}'s are monotonic, the subarc ax_{v_m} of T_x is contained within $S - x_{v_n} - y_{v_n}$ as are also N_{v_m} and U. From the preceding paragraph $N \subset S - x_i - y_i$. Likewise, the subarc pb of T_x is also contained in $S - x_i - y_i$. Hence, $G = ax_{v_m} + N_{v_m} + U + N + pb$ lies within $S - x_{v_n} - y_{v_n}$. But this is impossible since G is a connected set containing both a and b while the pair (x_{v_n}, y_{v_n}) separates a and b.

Thus the theorem is established that $L + a + b$ is closed and compact. The assumption need not be made that S has no cut point in general but merely that no single point x separates a and b. Under this latter assumption the arcs T_x and T_y exist.

PART II

The second part of this paper treats the following theorem:

If G is any collection of closed, mutually exclusive and non-separated sets X separating any two fixed points a and b of a connected and locally connected, separable metric space S, then the elements of G are ordered. Further, any infinite monotonic sub-collection (X_i) of G is convergent and has a non-vacuous limit M which separates a and b if $M \subset S-a-b$.*

DEFINITION. By *non-separated* is meant that if X_i and X_j are any two elements of G and $S_{a_i}+S_{b_i}$ is a separation of $S-X_i$, the set X_j is contained entirely within S_{a_i} or S_{b_i}.

The *ordering* of G is defined as follows: X_i is said to precede X_j, $X_i \propto X_j$, if $X_j \subset S_{b_i}$. Some consequences of this definition are: either $X_i \propto X_j$ or $X_j \propto X_i$; if $X_i \propto X_j$, X_j does *not* precede X_i; if $X_i \propto X_j \propto X_k$, then $X_i \propto X_k$.

Suppose that (X_i) is any infinite monotonic plus collection of sets X_i, that is, if $S_{a_i}+S_{b_i}$ is a separation of $S-X_i$ then $\sum_{k=1}^{i-1} X_k \subset S_{a_i}$ while $\sum_{k=i+1}^{\infty} X_k \subset S_{b_i}$. From this it is easily seen that no point of the limit superior of (X_i) is contained in any S_{a_i} or X_i, for that point would then be a limit point of S_{b_i}. It will be shown first that the limit superior X of (X_i) is non-vacuous. If $S_a = \sum_1^\infty S_{a_i}$ and $S_b = \prod_1^\infty S_{b_i}$, it is easily seen that $S_a \cdot S_b = 0$, for otherwise some i would exist such that $S_{a_i} \cdot S_{b_i}$ would not be vacuous. Now $S = S_a + S_b$, for if z is a point of S, either z is a point of some $X_i \subset S_{a_{i+1}} \subset S_a$ or not. If not, either z is contained in every S_{b_i}, that is, $z \subset S_b$, or, since z is not contained in $\sum_1^\infty X_i$, some n exists such that $z \subset S_{a_n} \subset S_a$. Now S_{a_i} is an open set, for if a point $p \subset S_{a_i}$, since X_i is closed, a connected region R exists such that $p \subset R \subset S - X_i$. That is, $R \subset S_{a_i}$, and hence, since the sum of any number of open sets is again an open set, S_a is open. On the assumption that lim sup $(X_i) = X = 0$, no point of S_b is a limit point of $\sum_1^\infty X_i$. Thus, if p is a point of S_b, a connected region R exists such that $p \subset R \subset S - \sum_1^\infty X_i$. As p is contained in every S_{b_i}, it follows that R is also. Therefore, S_a and S_b are

* For references on the ordering of the elements of G see G. T. Whyburn, *Non-separated cuttings of connected point sets*, Transactions of this Society, vol. 33 (1931).

mutually exclusive open sets containing a and b respectively. But two mutually exclusive open sets are mutually separated, so that the assumption that $X=0$ leads to the contradiction that S is not connected.

The supposition that (X_i) is not convergent again leads to a contradiction. For if (X_i) is not convergent, an infinite subcollection (X_{n_i}) of the X_i's exists such that lim sup $(X_i)=X \neq$ lim sup $(X_{n_i})=N$. Choose the X_{n_i}'s such that they are monotonic and form $S_a=\sum_1^\infty S_{a_{n_i}}$ and $S_b=\prod_1^\infty S_{b_{n_i}}-N$. Just as before S_a and S_b are mutually exclusive open sets whose sum is $S-N$. However, this is impossible since $X-N\neq 0$ and is contained in S_b while $\sum_1^\infty X_i \subset S_a$ (given any X_i an X_{n_i} exists such that $X_i \propto X_{n_i}$, that is, $X_i \subset S_{a_{n_i}} \subset S_a$). Thus we see that the collection (X_i) is convergent.

Since every monotonic collection is either monotonic plus or monotonic minus, an interchange of a and b will take care of the negative case. It merely remains to show that the limit M of (X_i) separates a and b if $M \subset S-a-b$. Assuming that the collection (X_i) is monotonic plus, and forming as before $S_a=\sum_1^\infty S_{a_i}$ and $S_b=\prod_1^\infty S_{b_i}-M$, we see that the sets S_a and S_b, being mutually exclusive open sets whose sum is $S-M$, form a separation of $S-M$. Also, as neither a nor b was contained in M, $a \subset S_a$ and $b \subset S_b$.

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