ON THE ZEROS OF CERTAIN POLYNOMIALS RELATED TO JACOBI AND LAGUERRE POLYNOMIALS*

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1. *Introduction.* We consider the polynomials defined as follows:

$$(1)\qquad J_n(x, \alpha, \beta) \equiv x^{1-\alpha}(1 - x)^{1-\beta}\frac{d^n}{dx^n}\left[x^{n+\alpha-1}(1 - x)^{n+\beta-1}\right],$$

$$(2)\qquad L_n(x, \alpha) \equiv x^{1-\alpha}e^x \frac{d^n}{dx^n}\left[e^{-x}x^{n+\alpha-1}\right],$$

where α and β are arbitrary real numbers. If $\alpha, \beta > 0$, they are known respectively as Jacobi and Laguerre polynomials, satisfying the following orthogonality relations:

$$\int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}J_m(x)J_n(x)dx = 0,$$

$$\int_0^\infty e^{-x}x^{\alpha-1}L_n(x)L_m(x)dx = 0,$$

$$(\alpha, \beta > 0; m, n = 0, 1, \cdots ; m \neq n).$$

From these relations it can be shown that all the zeros of the functions $J_x(x, \alpha, \beta)$ and $L_n(x, \alpha)$ are real, distinct, and lie respectively inside $(0, 1)$, $(0, \infty)$.

The following differential equations are also well known:

$$(3)\qquad x(1 - x)J_n''(x, \alpha, \beta) + \{\alpha - (\alpha + \beta)x\}J_n'(x, \alpha, \beta) + n(n - 1 + \alpha + \beta)J_n = 0, \qquad (\alpha, \beta > 0),$$

$$(4)\qquad xL_n''(x, \alpha) + (\alpha - x)L_n'(x, \alpha) + nL_n(x, \alpha) = 0.$$

Since (3) and (4) represent identical relations between the coefficients of $J_n(x, \alpha, \beta)$ and $L_n(x, \alpha)$ respectively which are polynomials in α, β or in α respectively, we conclude that the differential equations still hold, if $\alpha, \beta \leqq 0$.

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The object of this paper is to study the nature of the zeros of these polynomials when $\alpha, \beta \leqq 0$. In this case the orthogonality relations do not hold since the integrals involved do not exist. Consequently, the aforesaid conclusion about the zeros also fails. M. Fujiwara* has shown that if p and q are positive integers such that

$$0<\alpha+p<1, \qquad 0<\beta+q<1,$$

then $J_n(x, \alpha, \beta)$ has at least $n-p-q$ zeros in $(0, 1)$.

In what follows these results have been improved and given in a more precise form (Theorem 2) and similar results derived for $L_n(x, \alpha)$ (Theorem 1).

2. *On the Zeros of* $L_n(x, \alpha)$ *for* $\alpha \leqq 0$.

THEOREM 1. (i) *If p is a positive integer such that* $0<\alpha+p\leqq 1$, $L_n(x, \alpha)$ *for* $n \geqq p$, *has exactly* $n-p$ *zeros inside* $(0, \infty)$; (ii) *moreover, if* $\alpha+p=1$, $L_n(x, \alpha)$ *has an additional zero at* $x=0$ *of multiplicity* p.

PROOF. CASE 1. $0<\alpha+p<1$. First, by applying Fujiwara's method, we show that $L_n(x, \alpha)$ has at least $n-p$ zeros inside $(0, \infty)$. By (2) we write

$$x^{\alpha-1}e^{-x}L_n(x, \alpha) = \frac{d^n\psi}{dx^n}, \quad (\psi(x) = x^{n+\alpha-1}e^{-x}),$$

$$\int_0^\infty x^{\alpha+p-1}e^{-x}L_n(x, \alpha)x^m dx = \int_0^\infty x^{m+p}\frac{d^n\psi}{dx^n}dx.$$

(These two integrals exist for $\alpha+p-1>0$.) Furthermore, if $n>m+p$, integration by parts shows at once that the right-hand member vanishes. Hence

$$(5) \quad \int_0^\infty x^{\alpha+p-1}e^{-x}L_n(x, \alpha)x^m dx = 0, (m = 0, 1, \cdots, n-p-1).$$

Suppose, first, that $L_n(x, \alpha)$ has $r(<n-p)$ zeros in $(0, \infty)$:

$$\alpha_1, \alpha_2, \cdots, \alpha_r.$$

Then

* M. Fujiwara, *On the zeros of Jacobi's polynomials*, Japanese Journal of Mathematics, vol. 2 (1925), pp. 1–2.

$$L_n(x, \alpha) = (x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_r)P_{n-r}(x) \equiv R(x)P_{n-r}(x),$$

and (see (5)),

$$\int_0^\infty x^{\alpha+p-1}e^{-x}R^2(x)P_{n-r}(x)dx = 0,$$

which is impossible, since $P_{n-r}(x)$ does not change sign in $(0, \infty)$. Consequently

$$r \geqq n - p. \tag{6}$$

Secondly, we show that

$$r \leqq n - p.$$

Write

$$L_n(x, \alpha) = \sum_{i=0}^{n} \beta_i x^i.$$

Substituting in (4), we have

$$(i + 1)(\alpha + i)\beta_{i+1} = (i - n)\beta_i, \quad (i = 0, 1, \cdots, n - 1). \tag{7}$$

Since $0<\alpha+p<1$,

$$\alpha + i < 0 \quad \text{for} \quad 0 \leqq i \leqq p - 1;$$

$$\alpha + i > 0 \quad \text{for} \quad p \leqq i \leqq n.$$

Thus, $\beta_0, \beta_1, \cdots, \beta_p$ have like signs, $\beta_p, \beta_{p+1}, \cdots, \beta_n$ have alternate signs, and the sequence $\{\beta_i\}$, $(i=0, 1, \cdots, n)$, present exactly $n-p$ variations in sign. Hence, by Descartes' rule, $L_n(x, \alpha)$ has at most $n-p$ zeros in $(0, \infty)$, which, combined with (6), yields the desired conclusion, $r = n-p$.

CASE 2. $\alpha+p=1$. From (7) we have

$$\beta_0 = \beta_1 = \cdots = \beta_{p-1} = 0; \qquad \beta_p \neq 0.$$

Thus, $L_n(x, \alpha)$ has a zero of multiplicity p at $x=0$.

To show that the remaining zeros lie inside $(0, \infty)$, we write (see (5))

$$L_n(x, \alpha) \equiv R_{n-p}(x, \alpha)x^p; \quad \int_0^\infty x^{\alpha+2p-1}e^{-x}R_{n-p}(x, \alpha)x^m dx = 0,$$

$$(m = 0, 1, \cdots, n - p - 1).$$

Employing a similar argument to that used in Case 1, we conclude that $R_{n-p}(x, \alpha)$ has at least $n-p$ zeros inside $(0, \infty)$ and therefore exactly $n-p$ such zeros, since it is a polynomial of degree $n-p$.

3. *On the Zeros of* $J_n(x, \alpha, \beta)$ *for* $\alpha, \beta \leqq 0$.

THEOREM 2. (i) *If* p *and* q *are positive integers such that* $0<\alpha+p\leqq 1$, $0<\beta+q\leqq 1$, *then* $J_n(x, \alpha, \beta)$ *for* $n\leqq p+q+1$ *has exactly* $n-p-q$ *zeros inside* (0, 1). (ii) *If* $\alpha+p=1$, $J_n(x, \alpha, \beta)$ *has an additional zero of multiplicity* p *at* $x=0$; *if* $\beta+q=1$, $J_n(x, \alpha, \beta)$ *has a zero of multiplicity* q *at* $x=1$.

PROOF. CASE 1. $0<\alpha+p<1$; $0<\beta+q<1$. In view of M. Fujiwara's results, it is sufficient to show that the number of zeros of $J_n(x, \alpha, \beta)$ inside (0, 1) can not exceed $n-p-q$. This will be done in several steps.

First, we shall show that $J_n(x, \alpha+1, \beta)$ has at least one more zero inside (0, 1) than $J_n(x, \alpha, \beta)$. We get, making use of (1) and of the identity

$$\frac{d^n}{dx^n}[\psi x] \equiv x\frac{d^n\psi}{dx^n} + n\frac{d^{n-1}\psi}{dx^{n-1}},$$

$$(8)\; J_n(x, \alpha+1, \beta) = J_n(x, \alpha, \beta) + nx^{-\alpha}(1-x)^{-\beta+1}\frac{d^{n-1}}{dx^{n-1}}\phi(x),$$

$$(\phi(x) = x^{n+\alpha-1}(1-x)^{n+\beta-1}).$$

Employing the abbreviated notation

$$J_n(x, \alpha+1, \beta) - J_n(x, \alpha, \beta) \equiv T_n(\alpha)$$

and differentiating (8), we get, making again use of (1),

$$(9)\; n(1-x)J_n(x,\alpha,\beta) = [\alpha-(\alpha+\beta-1)x]T_n(\alpha) + x(1-x)T_n'(\alpha).$$

Differentiating (9) and using (3) written for $J_n(x, \alpha, \beta)$ and for $J_n(x, \alpha+1, \beta)$, we find

$$(10)\qquad (n-1+\alpha+\beta)[J_n(x, \alpha+1, \beta) - J_n(x, \alpha, \beta)] = (x-1)J_n'(x, \alpha, \beta).$$

We note that, if $n\geqq p+q+1$, then $n-1+\alpha+\beta>0$.

Let x_i and $x_{i+1}(>x_i)$ be two consecutive zeros of $J_n(x, \alpha, \beta)$ inside (0, 1). Then, comparing the signs of $J_n(x, \alpha, \beta)$ and of $J_n(x, \alpha+1, \beta)$ in (10) for $x=x_i$, and x_{i+1}, we conclude that there exists at least one zero of $J_n(x, \alpha+1, \beta)$ between x_i and x_{i+1}.

Next, if x_k be the right-most zero of $J_n(x, \alpha, \beta)$ inside (0, 1), we can show that there exists a zero of $J_n(x, \alpha+1, \beta)$ inside

$(x_k, 1)$. In fact, $J_n(1, \alpha, \beta) \neq 0$ (as we shall show later), say >0; hence, since

$$J_n'(x_k, \alpha, \beta) > 0, \qquad J_n(1, \alpha+1, \beta) > 0,$$

it follows that

$$J_n(x_k, \alpha+1, \beta) < 0$$

by (10).

In a similar fashion, if x_1 is the left-most zero of $J_n(x, \alpha, \beta)$ inside (0, 1), there exists a zero of $J_n(x, \alpha+1, \beta)$ inside $(0, x_1)$.

Combining the above results, we conclude that $J_n(x, \alpha+1, \beta)$ has at least one more zero; hence $J_n(x,\alpha+p, \beta)$ has at least p more zeros inside (0, 1) than $J_n(x, \alpha, \beta)$.

Consider now $J_n(x, \beta, \alpha+p)$. The obvious relation

$$J_n(x, \beta, \alpha + p) = (-1)^n J_n(1 - x, \alpha + p, \beta) \tag{11}$$

shows that $J_n(x, \beta, \alpha+p)$ has the same number of zeros inside (0, 1) as $J_n(x, \alpha+p, \beta)$. We come now to the final step in our proof.

Suppose $J_n(x, \alpha, \beta)$ has $n-p-q+k$ zeros inside (0, 1), where $k>0$. By the preceding argument $J_n(x, \alpha+p, \beta)$ and therefore $J_n(x, \beta, \alpha+p)$ have each at least $n-q+k$ zeros inside (0, 1). Repeating the argument, we see that $J_n(x, \beta+q, \alpha+p)$ has at least $n+k>n$ zeros inside (0, 1), which is impossible if $k>0$. Consequently, $k=0$, and our theorem is thus proved for Case 1.

We can easily prove what was tacitly assumed in the above argument, that $J_n(x, \alpha, \beta)$ has no multiple zeros inside (0, 1). Suppose $J_n(x, \alpha, \beta)$ has a multiple zero at x_i, so that

$$J_n(x_i, \alpha, \beta) = J_n'(x_i, \alpha, \beta) = 0;$$

from (3)

$$J_n''(x_i, \alpha, \beta) = J_n'''(x_i, \alpha, \beta) = \cdots = 0.$$

Another tacit assumption that $J_n(x, \alpha, \beta) \neq 0$ for $x=0, 1$ will be revealed in the discussion below.

REMARK. The same results hold, if $0<\alpha+p<1$ and $\beta>0$ (here $q=0$) or if $0<\beta+q<1$ and $\alpha>0$ (here $p=0$).

CASE 2. $\alpha+p=1$, $0<p+q<1$. Writing

$$J_n(x, \alpha, \beta) = \sum_{i=0}^{n} \gamma_i x^i$$

and substituting in (3), we obtain

$$\{n(n-1+\alpha+\beta)-i(i-1+\alpha+\beta)\}\gamma_i+(i+1)(\alpha+i)\gamma_{i+1}=0,$$
$$(i=0,1,\cdots,n-1).$$

Hence

$$\gamma_0=\gamma_1=\cdots=\gamma_{p-1}=0;\qquad \gamma_p\neq 0$$

(since $\alpha+p-1=0$), which shows that $x=0$ is a zero of multiplicity p of $J_n(x, \alpha, \beta)$, so that

$$J_n(x,\alpha,\beta)\equiv x^p R_{n-p}(x,\alpha,\beta).$$

In the same manner, as we showed for $L_n(x, \alpha)$, we can show that $R_{n-p}(x, \alpha, \beta)$ has at least $n-p-q$ zeros inside (0, 1). To find an upper limit for the number of these zeros, we substitute in (10)

$$J_n(x,\alpha,\beta)\equiv R_{n-p}(x,\alpha,\beta)x^p,$$
$$J_n(x,\alpha+1,\beta)\equiv R_{n-p+1}(x,\alpha+1,\beta)x^{p-1},$$

and obtain

$$(12)\quad (n-1+\alpha+\beta)R_{n-p+1}(x,\alpha+1,\beta)=x(x-1)R'_{n-p}(x,\alpha,\beta)$$
$$+\left[(n+p-1+\alpha+\beta)x-p\right]R_{n-p}(x,\alpha,\beta).$$

By an argument similar to that given before, (12) shows that $R_{n-p+1}(x, \alpha+1, \beta)$ has at least one more zero inside (0, 1) than $R_{n-p}(x, \alpha, \beta)$. Suppose now $R_{n-p}(x, \alpha, \beta)$ has $n-p-q+k$ zeros inside (0, 1), where $k>0$. Then $R_n(x, \alpha+p, \beta)$ has at least $n-q+k$ zeros inside (0, 1). But

$$R_n(x,\alpha+p,\beta)\equiv J_n(x,\alpha+p,\beta)$$

has exactly $n-q$ zeros inside (0, 1), as was shown in Case 1. Thus, $k=0$, and Theorem 2 is established for Case 2.

CASE 3. $0<\alpha+p<1, \beta+q=1$. From the above argument (see (11)) we can state immediately that $J_n(x, \alpha, \beta)$ has a zero at $x=1$ of multiplicity q, and exactly $n-p-q$ zeros inside (0, 1).

CASE 4. $\alpha+p=\beta+q=1$. It follows from Cases 2 and 3 that $J_n(x, \alpha, \beta)$ has zeros at $x=0$, 1 of multiplicity p, q respectively. Writing

$$J_n(x,\ \alpha,\ \beta)\equiv x^p(1-x)^q R_{n-p-q}(x,\ \alpha,\ \beta)$$

and applying M. Fujiwara's method to $R_{n-p-q}(x, \alpha, \beta)$, we readily show that R_{n-p-q} (x, α, β) has at least $n-p-q$ zeros inside (0, 1); hence, being of degree $n-p-q$, it has exactly $n-p-q$ such zeros.

4. *Remarks*. (i) The fact that $L_n(x, \alpha)$ has exactly $n-p$ zeros inside $(0, \infty)$ also follows by considering it as a limiting case of $J_n(x, \alpha, \beta)$. Suppose $\beta > 0$, $0 < \alpha + p \leqq 1$, and consider the transformation $x_1 = \beta x$. We know that the polynomial

$$\bar{J}_n(x_1, \alpha, \beta) = J_n(x_1/\beta, \alpha, \beta)$$

has exactly $h-p$ zeros inside $(0, \beta)$. On the other hand, by (1),

$$\bar{J}_n(x_1, \alpha, \beta) = x_1^{-\alpha}\left(1 - \frac{x_1}{\beta}\right)^{-\beta} \frac{d^n}{dx_1^n}\left[x_1^{n+\alpha}\left(1 - \frac{x_1}{\beta}\right)^{n+\beta}\right],$$

and since

$$\frac{d^i}{dx^i}\left[x^{h+\alpha}\left(1 - \frac{x}{\beta}\right)^{k+\beta}\right] \underset{\beta\to\infty}{\longrightarrow} \frac{d^i}{dx^i}\left[x^{h+\alpha}e^{-x}\right], \ (i, h, k = 0, 1, \cdots),$$

it follows from (2) that

$$\lim_{\beta\to\infty} \bar{J}_n(x_1, \alpha, \beta) = L_n(x, \alpha).$$

(ii) From the argument employed in Section 2, we conclude that inside (0, 1) the zeros of $J_n(x, \alpha+1, \beta)$ separate those of $J_n(x, \alpha, \beta)$ and conversely. The same is true of $J_n(x, \alpha, \beta)$ and $J_n(x, \alpha, \beta+1)$ and of $L_n(x, \alpha)$ and $L_n(x, \alpha+1)$, inside (0, 1), $(0, \infty)$, respectively.

(iii) The results of Section 2 evidently hold for any finite interval (a, b), the polynomials $J_n(x, \alpha, \beta)$ being defined as follows:

$$J_n(x, \alpha, \beta) = (x-a)^{1-\alpha}(b-x)^{1-\beta}\frac{d^n}{dx^n}\left[(x-a)^{n+\alpha-1}(b-x)^{n+\beta-1}\right].$$

(iv) The aforesaid property of the zeros of the orthogonal Laguerre and Jacobi polynomials $(\alpha, \beta > 0)$, that they lie inside $(0, \infty)$, (0, 1) respectively, follows at once from Theorems 1 and 2, if we make there $p=0$, $q=0$.

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