DICKSON ON THEORY OF NUMBERS

Studies in the Theory of Numbers. By L. E. Dickson. The University of Chicago Press, 1930. 10+230 pp.

More than a hundred years have elapsed since Gauss laid the foundation of the theory of ternary quadratic forms, developed by him only so far as was necessary for his immediate purposes. At the end of that section of *The Disquisitiones* dealing with ternary forms, Gauss points out for future investigators an immense field for research; namely, the arithmetical investigation of forms of higher degrees as well as of quadratic forms with more than two variables.

Despite the efforts of the greatest mathematicians up to our day, very little has been achieved in regard to forms of higher degrees, even in the binary case. On the contrary, the development of the theory of quadratic forms with three and more variables, in the hands of men like Eisenstein, Hermite, Smith, Minkowski, A. Meyer, and Voronoi, not to speak of other important contributors in this field, was crowned with considerable success. Their work is scattered in different publications and therefore is not easily accessible. Moreover, due to the natural complexity of the matter, it needs careful revision, since even the greatest among mathematicians are not proof against errors, which are sometimes slight and sometimes more serious, as the case may be.

The first attempt to present the theory of quadratic forms with three or more variables in its essential parts was made by Bachmann in two valuable volumes published in 1898 and 1923. Paying all due respect to an easy presentation of difficult matters and to the variety of subjects included in Bachmann's work, one cannot refrain from feeling that there still remains much to be desired. The main criticism of Bachmann's work, in the opinion of the reviewer, is that it lacks the critical revision of results and the proofs found in the original papers; so it often happens that defects for which even authoritative investigators should be held responsible are reproduced in Bachmann's book without change. One must be always on guard when trying to use Bachmann's *Arithmetik der Quadratischen Formen* as a reference book in his own research.

Under such circumstances, those who cultivate the field of quadratic forms —and their number, fortunately, is increasing—cannot but welcome the valuable volume under review. The rather indefinite title does not reveal that the book deals almost exclusively with questions pertaining to the arithmetical theory of ternary and quaternary quadratic forms. It is by no means a treatise on quadratic forms, but a presentation of some important selected topics of which Professor Dickson has made a profound study, and which he has enriched by valuable personal contributions. The reviewer and many others would certainly like to see a complete treatise on quadratic forms of the same high qualities as Professor Dickson's book, but it seems the time is not ripe for such an arduous enterprise, inasmuch as many important parts in the theory of quadratic forms are still in a state of infancy; for instance, the reduction theory of the indefinite (even ternary) forms.

Although limited in scope, Professor Dickson's book presents exactly what we miss in Bachmann's book: the carefully revised work of his predecessors, with consequent perfect reliability; many original results; and, finally, the concise and yet clear exposition so characteristic of the author.

The book is divided into three loosely connected parts. The first begins with the explanation of the fundamental properties of general quadratic forms, and is followed by an excellent introduction to the theory of ternary forms. Incidentally, in the second chapter Professor Dickson proves that an indefinite ternary quadratic form representing all integers (he calls it a *universal form*) must represent 0 for integral values of variables not vanishing simultaneously. The main object of the first part, however, is a thorough revision and completion of an extensive theory developed by Arnold Meyer in several papers which are by no means easy to read. Eisenstein once made a rather vague remark that, for indefinite ternary quadratic forms with relatively prime odd invariants, each genus contains one class. Meyer tried to prove this remark, and in general to find the number of classes contained in each genus under more general conditions. In the course of a rather complicated investigation, Meyer, it seems, did not escape making erroneous statements, and we must surely feel grateful to Professor Dickson for putting this important theory on a sound basis. In the final chapter of the first part Professor Dickson applies the previously obtained results to the proof of an interesting statement, also due to Meyer; namely, that an arbitrary indefinite form with more than four variables is a zero form. The proof developed here is a first complete proof of this theorem on the lines outlined by Meyer. A closely related question concerning the conditions of solvability of the diophantine equation $ax^2+by^2+cz^2+dt^2=0$ is also treated in the same chapter. One regrets that no mention is made of a recent work by H. Hasse, which seems to offer a more genuine approach to questions of this kind.

The second part of the book especially appeals to the sentimental feeling of the reviewer, since it deals with questions studied by the late Professor A. A. Markoff, whom he has the right to revere as his teacher. Korkine, who, together with Zolotareff, published extremely beautiful investigations on precise limits of minima of definite forms, suggested to Markoff that he consider similar problems for indefinite quadratic forms. Markoff, who then was only twenty-two years of age, in a short time attained most startling results in the case of binary forms, and published them first in the Mathematische Annalen, and later in more developed form as a thesis presented to the physico-mathematical section of the University of St. Petersburg in 1880. Many years later Markoff extended his work to ternary and quaternary forms, but here the results obtained by him are not as complete as in the case of binary forms.

Such are the questions dealt with in the second part of the *Studies*. Professor Dickson gives an elaborate exposition of Markoff's results in the binary case and rectifies and completes his investigations in the ternary case. The author then proceeds to explain new results obtained by Dr. A. Oppenheim for quaternary forms. It seems, however, that neither Professor Dickson nor Dr. Oppenheim was aware of a Russian paper by Markoff in the Bulletin de l'Académie Impérial des Sciences de St. Petersbourg, 1902, where the following result can be found: For indefinite quaternary forms of determinant D the precise limit of the minimum is $(4|D|/7)^{1/4}$ and this limit is attained for the forms equivalent to $\phi_0=(4|D|/7)^{1/4}[(x-\frac{1}{2}t)^2+(y-\frac{1}{2}t)^2+(z-\frac{1}{2}t)^2-7/4t^2]$.

The precise minimum for the remaining forms is $(4|D|/9)^{1/4}$ and is attained in the class of forms equivalent to

$$\phi_1 = (4|D|/9)^{1/4}(x^2+xy+y^2-2z^2-2zt-2t^2).$$

Any form not equivalent to ϕ_0 or ϕ_1 has a minimum less than $(4|D|/9)^{1/4}$.

The concluding chapter of the second part of *Studies* deals with tabulation of indefinite ternary quadratic forms not representing 0 and contains a table of non-equivalent forms with determinants not greater than 80 (with the exception of the determinant 68 where the equivalence of two forms $x^2-3y^2-2yz-23z^2$ and $x^2-7y^2-6yz-11z^2$ could not be decided).

A similar table extended to all determinants not greater than 50 was published earlier by Markoff with an important additional feature; namely, it includes for each form a set of linear homogeneous inequalities which define a domain surely containing only a limited number of representations of any integer by the corresponding form. It seems to the reviewer that the general proof that linear inequalities of such nature exist for any indefinite ternary form (not zero form) would be of the utmost importance and well worth investigation.

Passing to the third and last part of the *Studies*, which is devoted to miscellaneous investigations, we first call attention to Chapter 12, where the question of universal forms is resumed and brought to a solution in the case of ternary and quaternary forms.

The preceding chapter gives a very detailed elaboration of the reduction theory of positive ternary forms, in general based on Dirichlet's geometrical method, with some modifications intended to define reduced forms essentially in Eisenstein's manner. This is followed by a table of reduced positive ternary quadratic forms with determinants not exceeding 50. While it is very pleasant to have conditions whereby the reduced forms are uniquely defined, it seems that the approach to this problem by characterizing the unique reduced parallelipipedon is not the best. It is more natural to define a fundamental domain *in the space of the coefficients* of the form, as did Minkowski and, after him, Voronoi, in their profound investigations of the reduction theory of general positive quadratic forms. Applied to the ternary case, these general considerations lead easily to a perfect and elegant geometric solution of the reduction problem.

At the very end of Chapter 11, Professor Dickson gives the result due to Korkine and Zolotareff concerning the precise limit of minima of positive quaternary forms. The proof given here is unfortunately applicable only to quaternary forms, and we regret that no mention is made of the important later papers by Korkine and Zolotareff, whose work was continued first by Voronoi, and in recent times with great success by Professor H. F. Blichfeldt.

The last chapter of the book has little to do with the arithmetical theory of quadratic forms except in a remote sense. This chapter, prepared by Dr. Oppenheim, presents an elegant exposition of G. H. Hardy's analytical proof for the known formulas which express the number of representations of integers by sums of not more than eight squares.

In conclusion, we repeat that *Studies in the Theory of Numbers* is certainly a highly important contribution to the literature on quadratic forms, and that it should be consulted by everyone interested in this fascinating field of research.

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