To validate the process, proceed from $t=2$ by mathematical induction. For $t=2$, we consider $f(x_1, x_2)$ as a function of $x_1$, keep $x_2$ fixed, and apply the summation formulas of §2. In each term of the result we then consider $f(x_1, x_2)$ as a function of $x_2$ and apply §2.

In exactly the same way multiple summations equivalent to symbolic products (as above) of any number of factors of one or more of the types giving the explicit forms of the function $f_{\xi ps}(n)$, $(\xi=\beta, \gamma, \eta, \rho)$, in §2 can be written out as functions of the upper limits of the summations.

By the method of proof in §§1, 2, it follows that these formulas remain true under linear transformations of the arguments of the entire functions. The like does not hold for non-linear transformations, as the product of two or more umbrae is undefined.

CALIFORNIA INSTITUTE OF TECHNOLOGY

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# A LOGICAL EXPANSION IN MATHEMATICS*

BY HASSLER WHITNEY†

1. *Introduction*. Suppose we have a finite set of objects, (for instance, books on a table), each of which either has or has not a certain given property $A$ (say of being red). Let $n$, or $n(1)$, be the total number of objects, $n(A)$ the number with the property $A$, and $n(\bar{A})$ the number without the property $A$ (with the property not-$A$ or $\bar{A}$). Then obviously

$$n(\bar{A}) = n - n(A). \tag{1}$$

Similarly, if $n(A\ B)$ denote the number with both properties $A$ and $B$, and $n(\bar{A}\ \bar{B})$ the number with neither property, that is, with both properties not-$A$ and not-$B$, then

$$n(\bar{A}\bar{B}) = n - n(A) - n(B) + n(AB), \tag{2}$$

which is easily seen to be true.

The extension of these formulas to the general case where any number of properties are considered is quite simple, and is well

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† National Research Fellow.

known to logicians. It should be better known to mathematicians also; we give in this paper several applications which show its usefulness.

The notation is that used in a paper by the author, *Characteristic functions and the algebra of logic*,* which we shall refer to as C. F. It should cause no confusion if we use the same symbols for characteristic functions, sets, and properties. The paper named is not essential for an understanding of the present paper.

2. *The Logical Expansion*. We prove now the general formula

$$\begin{aligned} n(\bar{A}_1\bar{A}_2\cdots\bar{A}_m) &= n - [n(A_1) + n(A_2) + \cdots + n(A_m)] \\ &+ [n(A_1A_2) + n(A_1A_3) + \cdots + n(A_{m-1}A_m)] \\ &- [n(A_1A_2A_3) + \cdots ] + \cdots + (-1)^m n(A_1A_2\cdots A_m), \end{aligned} \tag{3}$$

which gives the number of objects *without* certain properties in terms of the numbers of objects *with* various of these properties. The formula is exactly what we should get if we multiplied out the expression in brackets in $n[(1-A_1)(1-A_2)\cdots(1-A_m)]$ by ordinary algebra and applied the general formulas

$$n(F+G) = n(F) + n(G),\ n(-F) = -n(F).$$

We assume the formula holds if there are two properties involved, and shall prove it for the case that three properties are involved. The proof obviously holds if the numbers two and three are replaced by $i$ and $i+1$, and hence the formula is true in general, by mathematical induction.

Consider the objects counted in $n(\bar{A}_1\bar{A}_2)$, that is, those with neither of the properties $A_1$ and $A_2$. We wish to know how many of these have the property $\bar{A}_3$. Applying (1) to this set, we have

$$n(\bar{A}_1\bar{A}_2\bar{A}_3) = n(\bar{A}_1\bar{A}_2) - n(\bar{A}_1\bar{A}_2A_3).$$

We know, by hypothesis, that

$$n(\bar{A}_1\bar{A}_2) = n - [n(A_1) + n(A_2)] + n(A_1A_2).$$

To find $n(\bar{A}_1\bar{A}_2A_3)$ we need merely consider those objects with the property $A_3$, and apply the expansion to this set. Thus

$$n(\bar{A}_1\bar{A}_2A_3) = n(A_3) - [n(A_1A_3) + n(A_2A_3)] + n(A_1A_2A_3).$$

* To appear in the Annals of Mathematics.

Hence

$$(4)\quad n(\bar{A}_1\bar{A}_2\bar{A}_3) = n - [n(A_1) + n(A_2) + n(A_3)] + [n(A_1A_2) + n(A_1A_3) + n(A_2A_3)] - n(A_1A_2A_3),$$

as required.

3. *The Measure of Characteristic Functions.* This section relates this paper to the paper C. F. We shall find a general class of formulas which contains the logical expansion as a special case, making use of *characteristic functions*. With each element $x$ of a set of $n$ objects $R$ we associate an integer $A(x)$, positive, negative, or zero. We define the *measure of A* by the equation

$$(5)\qquad n(A) = \sum_{x \text{ in } R} A(x).$$

Suppose $A$ were one for certain elements of $R$ (which elements form the set $A'$), and zero for the rest. Then $A$ is the characteristic function of $A'$, and $n(A)$ is just the number of elements in $A'$. In particular, if $A \equiv 1$, that is, $A'$ is $R$, then $n(A)$ is

$$(6)\qquad n(1) = n,$$

and if $A \equiv 0$, that is, $A'$ contains no elements, then $n(A)$ is

$$(7)\qquad n(0) = 0.$$

If $A$ and $B$ are any functions and $p$ and $q$ are any numbers, then

$$(8)\qquad \begin{aligned} n(pA + qB) &= \sum_{x \text{ in } R} [pA(x) + qB(x)] = p\sum A(x) + q\sum B(x) \\ &= pn(A) + qn(B). \end{aligned}$$

The logical expansion now follows at once if we expand $\bar{A}_1\bar{A}_2 \cdots \bar{A}_m$ into the second normal form as in C. F.*

4. *On Prime Numbers.* Let

$$p_1, p_2, \cdots, p_m$$

be any set of positive integers. We wish to find $N$, the number of numbers less than or equal to a given number $x$, which are not divisible by any of these numbers. If we let $R$ be the set of all numbers $\leq x$ and let $P_i$ be those divisible by $p_i$, then

* Note that, for characteristic functions, $\bar{A}_i = 1 - A_i$. We make this substitution, multiply out, and use (8).

$$
\begin{aligned}
N &= n(\overline{P}_1\overline{P}_2 \cdots \overline{P}_m) \\
&= n - [n(P_1) + n(P_2) + \cdots] + [n(P_1P_2) + \cdots] \qquad (9)\\
&\quad - \cdots + (-1)^m n(P_1P_2 \cdots P_m),
\end{aligned}
$$

by the logical expansion (where $n = x$). This expansion is fundamental in prime number theory.* Its importance lies in the fact that the terms of the form $n(P_{i_1}P_{i_2} \cdots P_{i_k})$ are easier to calculate directly than the desired quantity.

Suppose we wish to find the number $\phi(x)$ of prime numbers $\leqq x$. Let $p_1, p_2, \cdots, p_m$ be the primes $<\sqrt{x}$. Then the primes $\leqq x$ are just those numbers $\leqq x$ which are not divisible by any of the numbers $p_1, p_2, \cdots, p_m$, except for the number 1, together with the $m$ numbers $p_1, p_2, \cdots, p_m$. Thus

$$
\phi(x) = N + m - 1. \qquad (10)
$$

5. *A Problem in Probability.* Suppose a pack of cards is lying in a row on a table. If we lay out another pack of cards on top of these, what is the probability that no card of the second pack will lie on the same card of the first? If a pack contains $m$ cards, there are $m!$ arrangements of the cards in the second pack. If there are $N$ arrangements of the second pack such that no card of the second pack falls on the same card of the first, then the required probability is $p = N/m!$. If $n(A_i)$ denotes the number of arrangements such that the $i$th card of the second pack falls on the same card of the first, etc., then $N = n(\overline{A}_1 \cdot \overline{A}_2 \cdots \overline{A}_m)$. If we use the logical expansion, a typical term of the result is $(-1)^k n(A_{i_1} \cdot A_{i_2} \cdots A_{i_k})$. This is the number of arrangements in which the cards numbered $i_1, i_2, \cdots, i_k$ fall on the same cards of the first pack. These $k$ cards being fixed in position, there are $(m-k)!$ arrangements of the remaining cards. The term is thus $(-1)^k(m-k)!$. There are $\binom{m}{k}$ terms with $k$ factors, contributing together $(-1)^k\binom{m}{k}(m-k)! = (-1)^k m!/k!$. Summing over $k$ and dividing by $m!$, we have

$$
p = \sum_{k=0}^{m} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^m \frac{1}{m!},
$$

which is the sum of the first $m+1$ terms in the expansion of $1/e$.

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* See for instance E. Landau, *Primzahlen*, Leipzig, Tuebner, pp. 67 ff.

6. *On the Number of Ways of Coloring a Graph*. Consider any set of objects $a, b, c, \cdots, f$, and any set of pairs of these objects, as $ab, bd, \cdots, cf$. We call the whole collection a *graph*, containing the *vertices* $a, b, c, \cdots, f$, and the *arcs* $ab, bd, \cdots, cf$. It can be visualized simply by letting the vertices be points in space, and letting each arc be a curve joining the two vertices involved.

Suppose we have a fixed number $\lambda$ of colors at our disposal. Any way of assigning one of these colors to each vertex of the graph in such a way that any two vertices which are joined by an arc are of different colors, will be called an admissible coloring of the graph. We wish to find the number $M(\lambda)$ of admissible colorings, using $\lambda$ or fewer colors.

As a special case of this general problem we have the four-color map problem: To know if we can assign to each region of a map on a sphere one of four colors in such a way that no two regions with a common boundary are of the same color, that is, to see if $M(4)>0$. Our graph (which is just the dual graph of the map) is constructed by placing a vertex in each region of the map, and joining two vertices by an arc if the corresponding regions have a common boundary. We shall deduce a formula for the number $M(\lambda)$ of ways of coloring a graph due to Birkhoff.*

If there are $V$ vertices in the graph $G$, then there are $\lambda^V$ possible colorings, formed by giving each vertex in succession any one of the $\lambda$ colors. Let $R$ be this set of colorings. Let $A_{ab}$ denote those colorings with the property that $a$ and $b$ are of the same color, etc. Then the set of admissible colorings is

$$\overline{A}_{ab}\overline{A}_{bd}\cdots\overline{A}_{cf},$$

and the number of colorings is, if there are $E$ arcs in the graph,

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* *A determinant formula for the number of ways of coloring a map*, Annals of Mathematics, (2), vol. 14 (1912), pp. 42–46. The formula was discovered independently by the author by the method here given. See in this connection a paper by Birkhoff, *On the number of ways of coloring a map*, Proceedings of the Edinburgh Mathematical Society, (2), vol. 2 (1930), pp. 83–91; also a paper by the author, *The coloring of graphs*, to appear in the Annals of Mathematics. In regard to the four-color map problem, see references in Birkhoff's last mentioned paper, also a paper by the author, *A theorem on graphs*, Annals of Mathematics, (2), vol. 32 (1931), p. 379.

$$\begin{aligned} M(\lambda) &= n(\overline{A}_{ab}\overline{A}_{bd} \cdots \overline{A}_{cf}) \\ &= n - [n(A_{ab}) + n(A_{bd}) + \cdots + n(A_{cf})] \\ &\quad + [n(A_{ab}A_{bd}) + \cdots ] - \cdots \\ &\quad + (-1)^E n(A_{ab}A_{bd} \cdots A_{cf}). \end{aligned} \tag{11}$$

With each property $A_{ab}$ is associated an arc $ab$ of $G$. In the logical expansion, there is a term corresponding to every possible combination of the properties $A_{pq}$; with this combination we associate the corresponding arcs, forming a *subgraph* $H$ of $G$. In particular, the first term corresponds to the subgraph containing no arcs, and the last term corresponds to the whole of $G$. *We let $H$ contain all the vertices of $G$.*

Let us evaluate a typical term, such as $n(A_{ab}A_{ad} \cdots A_{ce})$. This is the number of ways of coloring $G$ in $\lambda$ or fewer colors in such a way that $a$ and $b$ are of the same color, $a$ and $d$ are of the same color, $\cdots$, $c$ and $e$ are of the same color. In the corresponding subgraph $H$, any two vertices that are joined by an arc must be of the same color, and thus all the vertices in a single connected piece in $H$ are of the same color. If there are $p$ connected pieces in $H$, the value of this term is therefore $\lambda^p$. If there are $s$ arcs in $H$, the sign of the term is $(-1)^s$. Thus

$$(-1)^s n(A_{ab}A_{ad} \cdots A_{ce}) = (-1)^s \lambda^p.$$

If there are $(p, s)$ (this is Birkhoff's symbol) subgraphs of $s$ arcs in $p$ connected pieces, the corresponding terms contribute to $M(\lambda)$ an amount $(-1)^s(p, s)\lambda^p$. Therefore, summing over all values of $p$ and $s$, we find the polynomial in $\lambda$:

$$M(\lambda) = \sum_{p,s}(-1)^s(p, s)\lambda^p. \tag{12}$$

Consider a subgraph $H$ of $G$ of $s$ arcs, in $p$ connected pieces. Let us define its rank $i$ and its nullity $j$ by the equations $i = V - p$, $j = s - i = s - V + p$. Then $p = V - i$, $s = i + j$, and putting $(p, s) = m_{V-p,s-V+p} = m_{ij}$, (thus, $m_{ij}$ is the number of subgraphs of $G$ of rank $i$, nullity $j$), we have

$$\begin{aligned} M(\lambda) &= \sum_{i,j}(-1)^{i+j} m_{ij}\lambda^{V-i} = \sum_i m_i \lambda^{V-i}, \\ m_i &= \sum_j (-1)^{i+j} m_{ij}. \end{aligned} \tag{13}$$

7. *The $m_i$ in Terms of the Broken Circuits of $G$.* We shall find in this section an interpretation of the coefficients of $M(\lambda)$ directly in terms of properties of the graph. Consider for instance the graph $G$ containing the vertices $a$, $b$, $c$, $d$ and the arcs $ab$, $ac$, $bc$, $bd$, $cd$, the arcs being given in this definite order. Make a list of the *circuits* in $G$, naming the arcs of a circuit in the order in which they occur above. Here, the circuits are $ab$, $ac$, $bc$, and $bc$, $bd$, $cd$, and $ab$, $ac$, $bd$, $cd$. From each circuit we now form the corresponding *broken circuit* by dropping out the last arc of the circuit. The broken circuits here are $ab$, $ac$, and $bc$, $bd$, and $ab$, $ac$, $bd$. Then *the number $(-1)^i m_i$ is the number of subgraphs of $G$ of $i$ arcs which do not contain all the arcs of any broken circuit.*

To show this, we arrange the broken circuits of $G$ in a definite order, where we put a broken circuit $P_i$ before a broken circuit $P_j$ if, in naming the arcs of $G$ one by one in the given order, all the arcs of $P_i$ are named before all those of $P_j$ are named, otherwise, the ordering is immaterial. Suppose there are $\sigma$ broken circuits, $P_1, P_2, \cdots, P_\sigma$. We now divide the subgraphs of $G$ into $\sigma+1$ sets (some of which may be empty), putting in the first set, $S_1$, all those subgraphs containing all the arcs of $P_1$; in the second, $S_2$, all those not containing $P_1$, but containing $P_2$; in the third, $S_3$, all those containing neither $P_1$ nor $P_2$, but containing $P_3$; $\cdots$ ; in the last set, $S_{\sigma+1}$, all those containing none of these broken circuits.

Consider now all the terms in (11) corresponding to the first set of subgraphs. Suppose $\alpha_1$ is the arc we dropped out of a circuit to form the first broken circuit $P_1$. To each subgraph in $S_1$ not containing $\alpha_1$ corresponds a subgraph in $S_1$ containing $\alpha_1$ and conversely, as $\alpha_1$ is not in $P_1$. The subgraphs of $S_1$, and hence the corresponding terms of (11), are thus paired off. But the two terms of each pair cancel. For let $H$ and $H'$ be the two corresponding subgraphs. If $H$ is in $p$ connected pieces, so is $H'$, as the arc $\alpha_1$ joins two vertices already connected by the broken circuit $P_1$. The terms each contribute $\lambda^p$ therefore; but they are of opposite sign, as $H'$ contains one more arc than $H$.

Consider now the terms corresponding to $S_2$ (if there are any such). If $\alpha_2$ is the arc dropped out in forming $P_2$, $\alpha_2$ is in neither $P_1$ nor $P_2$, on account of the way we have ordered the broken circuits. Thus to each subgraph in $S_2$ not containing $\alpha_2$ corresponds a subgraph in $S_2$ containing $\alpha_2$, and conversely. The corresponding terms of (11) are thus paired off, and they cancel, exactly as before.

Continuing, we cancel all terms in $S_3$, $S_4$, $\cdots$, $S_\sigma$. We are left only with terms in $S_{\sigma+1}$, that is, those corresponding to subgraphs not containing all the arcs of any broken circuit, and none of these have been canceled.

Consider any such term containing $i$ arcs. The corresponding subgraph $H$ contains no circuit, as it contains no broken circuit. If we build up $H$ arc by arc, each arc we add joins two vertices formerly not connected therefore, and the number of connected pieces is decreased by one each time. Thus the number of connected pieces in $H$ is $V-i$, and the corresponding term contributes $(-1)^i\lambda^{V-i}$ to $M(\lambda)$. If there are $l_i$ such subgraphs, they together contribute an amount $(-1)^i l_i\lambda^{V-i}$. Hence, summing over $i$, we have $P(\lambda)=\sum_i(-1)^i l_i\lambda^{V-i}$. Comparing with (13), we see that $l_i=(-1)^i m_i$, as required.

EXAMPLES. Let $G$ contain the vertices $a$, $b$, $c$, and the arcs $ab$, $ac$, $bc$. There is one broken circuit: $ab$, $ac$. There is one subgraph of no arcs, and $m_0=1$. There are three subgraphs of a single arc, and $-m_1=3$. There are three subgraphs of two arcs; but one of them contains the broken circuit, so $m_2=2$. The subgraph of three arcs contains the broken circuit. Hence, as $V=3$,

$$M(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2).$$

This is easily verified. For we can color $a$ in $\lambda$ ways; there are $\lambda-1$ colors left for $b$; there are now $\lambda-2$ colors left for $c$.

Let $G$ be the graph named at the beginning of this section. If a subgraph contains the last broken circuit, it contains the first also, so we can forget the last. We find

$$M(\lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda = \lambda(\lambda - 1)(\lambda - 2)^2,$$

which again is easily verified.

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