1. Introduction. Consider an algebra \((K, +, \times)\), such as ordinary real algebra, in which there are two elements “0” and “1” having the properties that, for any element \(a\),

\[
(1) \quad a + 0 = 0 + a = a, \quad a1 = 1a = a.
\]

Let

\[
(x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m)
\]

denote a unit-zero function with respect to the sequence of \(m\) elements, \(a_1, a_2, \ldots, a_m\) of \(K\), that is, a function \(f(x_1, x_2, \ldots, x_m)\) of \(m\) elements \(x_1, x_2, \ldots, x_m\) such that \(f = 1\) or \(0\), according as the equalities, \(x_i = a_i\) \((i = 1, 2, \ldots, m)\), all hold or do not all hold. Accordingly, \((x; a)\) will denote a unit-zero function with respect to \(a\), that is, a function \(f(x)\) such that \(f(x) = 1\) or \(0\), according as \(x = a\) or \(x \neq a\). Then the following propositions (2)–(4) evidently hold:

\[
(2) \quad (x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m) = (x_1; a_1)(x_2; a_2)\cdots(x_m; a_m);
\]

\[
(3) \quad a(x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m) = a \text{ or } 0,
\]

according as \(x_i = a_i\) \((i = 1, 2, \ldots, m)\), all hold or do not all hold;

\[
(4) \quad a(x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m)
\]

\[
+ b(x_1, x_2, \ldots, x_m; b_1, b_2, \ldots, b_m) = a, \text{ or } b, \text{ or } 0,
\]

according as \(x_i = a_i\) all hold, or \(x_i = b_i\) all hold, or neither \(x_i = a_i\) all hold nor \(x_i = b_i\) all hold, \((i = 1, 2, \ldots, m; a_i \neq b_i\) for some \(i)\).

In a previous paper† propositions (1)–(4) were made the basis of a method of obtaining arithmetic representations of arbitrary operations and relations in a finite class of elements. Since

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* Presented to the Society, April 11, 1931.

propositions (1)–(4) also hold when the symbols belong to Boolean algebra, the question naturally arises: To what extent can unit-zero functions be used analogously to obtain Boolean representations of arbitrary operations and relations? The object of the present paper is to answer this question.

2. Determination of Boolean Unit-Zero Algebras. The possibility of representing arbitrary operations and relations by unit-zero functions of an algebra hinges on the existence in this algebra of a unit-zero function for every sequence of \( m \) of its elements. Let us call an algebra which has a unit-zero function for every sequence of \( m \) of its elements a \textit{unit-zero algebra}. I proceed first to determine all Boolean unit-zero algebras.

This determination is made easy by noting at the outset that a unit-zero Boolean function must satisfy proposition (2) above and also that it must be single-valued. We therefore need to look only for Boolean unit-zero functions \( f(x) \) of a \textit{single} variable \( x \) of the form

\[
(x; a) = (1; a)x + (0; a)x'.
\]

From (5) we see, by putting \( a = 0, 1 \), that in a Boolean algebra of \textit{two} elements, \( x \) is the unit-zero function of \( x \) with respect to 1, and \( x' \) is the unit-zero function of \( x \) with respect to 0; in symbols,

\[
(x; 1) = x, \quad (x; 0) = x'.
\]

We have, then, that a \textit{two-element Boolean algebra is a unit-zero algebra, the unit-zero functions of one variable \( x \) being given by (6).}

By (2) and (6), \textit{all the unit-zero functions of a two-element Boolean algebra can be readily written down. Thus, the unit-zero functions of two variables \( x, y \) are given by}

\[
\begin{align*}
(x, y; 1, 1) &= xy, \quad (x, y; 1, 0) = xy', \\
(x, y; 0, 1) &= x'y, \quad (x, y; 0, 0) = x'y'.
\end{align*}
\]

In general, \textit{the unit-zero functions of \( m \) variables are the \( 2^m \) constituents in the normal development of 1 with respect to the \( m \) variables.}

* The usual Boolean notations are employed: \( a + b, ab, a', 0, 1 \) are respectively the sum of \( a \) and \( b \), the product of \( a \) and \( b \), the negative of \( a \), the zero element, the \textit{whole}. 

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Let us now consider a Boolean algebra $A$ of more than two elements. $A$ must have an element $e \neq 0, 1$. Suppose, first, that $A$ has a unit-zero function $f(x)$, of form (5), with respect to $e$. Then

(i) \[ f(e) = 1, f(0) = 0, f(1) = 0, \quad (e \neq 0, 1). \]

But (i) is inconsistent with (5). Hence, our algebra $A$ has no unit-zero function with respect to a sequence containing the element $e$.

Suppose, next, that the algebra $A$ has a unit-zero function $f(x)$, of form (5) with respect to 0. Then

(ii) \[ f(0) = 1, f(1) = 0, f(e) = 0, \quad (e \neq 0, 1). \]

Hence, by (5),

(iii) \[ f(x) = x', f(e) = 0, \quad (e \neq 0, 1). \]

But equations (iii) are inconsistent. Hence, our algebra $A$ has no unit-zero function with respect to a sequence containing the element 0.

Similarly, our algebra $A$ has no unit-zero function with respect to a sequence containing the element 1. Hence, a Boolean algebra of more than two elements has no unit-zero functions at all.

Our main result is, then, the following theorem.

**Theorem A.** The only Boolean unit-zero algebra is a two-element Boolean algebra.

3. Dual Considerations. By the Principle of Duality in Boolean algebras each of the foregoing propositions about unit-zero Boolean functions has a dual proposition corresponding to it. To state these duals, let me use the notion of zero-unit function (to be distinguished from unit-zero function). By a zero-unit function of $x_1, x_2, \cdots, x_m$ with respect to the sequence $a_1, a_2, \cdots, a_m$, symbolized by

\[ [x_1, x_2; \cdots, x_m; a_1, a_2, \cdots, a_m], \]

let us mean a function $f(x_1, x_2, \cdots, x_m)$ such that $f = 0$ or 1, according as $x_i = a_i, (i = 1, 2, \cdots, m)$, all hold or do not all hold. The duals of (2), (3), and (4) are, then, respectively (2'), (3'), and (4') following:
(2') \[ [x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m] = [x_1; a_1] + [x_2; a_2] + \cdots + [x_m; a_m]; \]
(3') \[ a + [x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m] = a \text{ or } 1, \]
according as \( x_i = a_i \) (\( i = 1, 2, \ldots, m \)), all hold or do not all hold;
(4') \[ \{ a + [x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_m] \} \cdot \{ b + [x_1, x_2, \ldots, x_m; b_1, b_2, \ldots, b_m] \} = a, \text{ or } b, \text{ or } 1, \]
according as \( x_i = a_i \) all hold, or \( x_i = b_i \) all hold, or neither \( x_i = a_i \) all hold nor \( x_i = b_i \) all hold, \((i = 1, 2, \ldots, m; a_i \neq b_i \text{ for some } i)\).

The dual of Theorem A is

**Theorem A'.** The only zero-unit Boolean algebra is a two-element Boolean algebra.

For a two-element Boolean algebra we have, further:

(6') \[ [x; 0] = x, \quad [x; 1] = x'; \]
(7') \[ [x, y; 0, 0] = x + y, \quad [x, y; 0, 1] = x + y', \quad [x, y; 1, 0] = x' + y, \quad [x, y; 1, 1] = x' + y'. \]

In general, the zero-unit functions of \( m \) variables are the \( 2^m \) factor-constituents in the dual normal development of 0 with respect to the \( m \) variables.

Propositions (2')–(7') will be used below in the representation of operations that do not satisfy the condition of closure.

4. **Representations.** It is now clear to what extent we can apply unit-zero Boolean functions in the representation of arbitrary operations and relations. From Theorem A, we have

**Theorem B.** A unit-zero Boolean representation of arbitrary operations and relations is possible when and only when the class consists of two elements.

For a two-element class \( K \), the theory of Boolean representation follows from propositions (2)–(7) and their duals. If we denote the two \( K \)-elements by the Boolean symbols 0, 1, the representations of all operations \( O \) and relations \( R \) in \( K \) are covered by the cases 1–3 following.

**Case 1.** \( O \) an \( m \)-ary operation satisfying the condition of closure. There is a \( K \)-element, 0 or 1, for every sequence \( e_1, e_2, \ldots, e_m \) taken from \( K \). Let the sequences to which 1 corresponds be
(i) $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im}; \alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{km}$.

The representation of $O$ is the Boolean function

$$\sum_{i=1}^{k} (x_1, x_2, \ldots, x_m; \alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im}).$$

**Case 2.** $O$ an $m$-ary operation not satisfying the closure condition. There are sequences in $K$ to which no $K$-elements correspond. Let these sequences be

(ii) $\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}; \beta_{k1}, \beta_{k2}, \ldots, \beta_{km}$.

Consider the operation $O'$ obtained from $O$ by assigning a $K$-element, 0 for convenience, to each of the sequences (ii). Let $\phi(x_1, x_2, \ldots, x_m)$, obtained as in Case 1, be the representation of $O'$. Then the representation of $O$ is the function

$$\phi(x_1, x_2, \ldots, x_m) + \sum_{i=1}^{k} 0/[x_1, x_2, \ldots, x_m; \beta_{i1}, \beta_{i2}, \ldots, \beta_{im}],$$

where $a/b$ means the unique $K$-element $q$ satisfying the condition $bq = a$. *

**Case 3.** $R$ an $m$-adic relation. Let the sequences which do not satisfy $R$ be

(iii) $\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{im}; \gamma_{k1}, \gamma_{k2}, \ldots, \gamma_{km}$.

Then the representation of $R$ is the Boolean equation

$$\sum_{i=1}^{k} (x_1, x_2, \ldots, x_m; \gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{im}) = 0.$$

Of course, by the Duality Principle, the theory of representation can be stated primarily in terms of zero-unit functions instead of unit-zero functions.

5. Illustrations. The following illustrations, one for each of the above three cases, will make the theory of representation quite clear.

**a.** Let $O$ be the operation defined by

* For a two-element Boolean algebra the quotient can be defined precisely as in ordinary algebra.

† Instead of 0, we can use 1 in (10), provided (i) are the sequences which do satisfy $R$. 

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Its representation is

(ii) \((x, y; 0, 0) + (x, y; 1, 1) = x'y' + xy\).

\(\beta\). Let \(O\) be the operation

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where the blanks indicate that there are no \(K\)-elements corresponding to the sequences 1, 0; 1, 1.

Consider the operation \(O'\) defined by

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By Case 1, the representation of \(O'\) is

(v) \(x'y\).

Hence, the representation of \(O\) is

(vi) \(x'y + 0/[x, y; 1, 0] + 0/[x, y; 1, 1] = x'y + 0/(x' + y) + 0/(x' + y')\).

\(\gamma\). Let \(R\) be a relation defined by

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where “+” indicates that \(R\) holds and “−” indicates that \(R\) does not hold. Its representation is the equation

(viii) \((x, y; 0, 0) + (x, y; 1, 1) = x'y' + xy = 0.\)

* For a complete set of Boolean representations of binary operations and dyadic relations in a two-element class, obtained from considerations other than the above, see my *Complete sets of representations of two-element algebras*, this Bulletin, vol. 30 (1924), pp. 24–30.