RESULTS AND PROBLEMS ABOUT \( n \)-WEBS
OF CURVES IN A PLANE

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If

\[
x^* = u(x, y), \quad y^* = v(x, y)
\]

is a topological mapping of the \( x, y \) plane, we call the function \( u \) topologically equivalent to \( x \) in this plane. Let us assume \( n \) such functions \( t_i(x, y), (i = 1, 2, \cdots, n) \), in the same simply connected domain \( D \). We have there \( n \) sheaves of curves \( t_i(x, y) = \text{const} \). We call this figure an \( n \)-web if two curves of different sheaves have not more than one point in common. We suppose the functions \( u_{ik}(t_i) \) to be continuous and strictly monotonic, so that for all pairs \( t_i \neq t_i' \), we have \( u_{ik}(t) \neq u_{ik}(t') \).

It seems to be interesting to study \( n \)-webs satisfying the condition that there are such functions \( u_{ik}(t_i) \) satisfying identically in \( D \) the relations

\[
\sum_{i=1}^{n} u_{ik}(t_i) = \text{const.}, \quad (k = 1, 2, 3, \cdots, m).
\]

We call these equations (1) linearly independent, if the identities

\[
\sum_{k=1}^{m} c_k u_{ik}(t_i) = \text{const.}
\]

imply for the constants \( c_k \) the trivial solution \( c_k = 0, (k = 1, 2, 3, \cdots, m) \). The following theorems hold.

**Theorem 1.** A 3-web satisfying one condition (1), \( (n = 3, m = 1) \), is topologically equivalent to the tangents of a curve of class 3 (irreducible or not).

This was essentially found by Graf and Sauer\(^\dagger\) in 1924. Howe and I\(^\ddagger\), in 1932, proved the following theorem.

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\(^\ddagger\) W. Blaschke and G. Howe, Hamburg Abhandlungen, vol. 9 (1932).
Theorem 2. A straight lined $n$-web satisfying (at least) one condition (1) is necessarily equivalent to the tangents of a curve of class $n$ ($n \geq 3$).

Our result contains Theorem 1 as a special case, because a 3-web satisfying the equation

$$u_1 + u_2 + u_3 = \text{const.}$$

is equivalent to a special straight lined 3-web (hexagonal web), as we see if we assume $u_1$ and $u_2$ as parallel coordinates.

Howe observed that the following Theorem 3 is equivalent to S. Lie’s results about the surfaces, which are translation surfaces in different ways.

Theorem 3. A 4-web satisfying 3 linearly independent relations ($n = 4$, $m = 3$) is equivalent to the tangents of a curve of class 4.

A geometric interpretation of one condition (1) for a 4-web has been given by Bose and myself.† Bol‡ discovered a short time ago the following result.

Theorem 4. The maximum number $m$ of linearly independent relations (1) for an $n$-web is

$$m = \frac{(n - 1)(n - 2)}{2}.$$  

Almost equivalent to a theorem of Reidemeister§ are the following.

Theorem 5. A 4-web satisfying 3 linearly independent relations (1) with $u_{ii} = 0$ is equivalent to 4 pencils of straight lines, no 3 of the 4 vertices on a straight line.

Theorem 6. A 4-web satisfying 3 relations (1) with $u_{ii} = 0$, ($i = 1, 2, 3$), only two of them linearly independent, admits a continuous one-parameter group, the $t_i = \text{const.}$ being paths.

But between the proofs of these theorems there is an essential difference Only Theorem 1 and the greater part of Theorems 5 and 6 are proved without any further restrictions for the functions $t_i$, $u_{ik}$. The proofs already known for Theorems 2–6

contain regularity restrictions. Therefore the first problem to be solved is the following one.

**Problem A.** Do the Theorems 2–5 remain valid without further regularity restrictions?

Another question unsolved as far as I know is the following one.

**Problem B.** To extend our Theorem 3 to n-webs.

These problems seem to be interesting because, for example, they contain a kind of real geometrical interpretation of Abel's theorem on algebraic curves.

Finally a few words about more dimensions. The questions about webs of surfaces

\[ S_i(x, y, z) = \text{const.} \]

in a 3-space can partially be reduced to our theorems on curve-webs in a 2-space. But if we consider sheaves of curves

\[ s_i(x, y, z) = \text{const.}, \quad t_i(x, y, z) = \text{const.} \]

in a 3-space we may ask, for example, the following question.

**Problem C.** How many essentially different relations

\[ u_1(s_1, t_1) + u_2(s_2, t_2) + u_3(s_3, t_3) = \text{const.} \]

*can exist for a 3-web of curves*

\[ s_i, t_i = \text{const.}, \quad (i = 1, 2, 3), \]

*in a 3-space?*

This seems to me to be one of the most promising fields of geometric research.

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