TRANSFORMATIONS ASSOCIATED WITH THE LINES OF A CUBIC, QUADRATIC, OR LINEAR COMPLEX*

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1. *Introduction.* In this paper it is shown that two equations bilinear in p_{ik} and x_i define an extensive type of cubic complex and also map the complex on the space (x) so that each line is mapped by a point on itself. The cubic complex of lines joining corresponding points of the general cubic involutorial transformation is included as a special case. The method is also applied to two known cases of the quadratic complex and the linear non-special and special complex.

2. *The Cubic Complex.* Let

$$\sum x_i f_i(p) = 0, \tag{1}$$

and

$$\sum x_i F_i(p) = 0, \qquad (i = 1, 2, 3, 4), \tag{2}$$

be two equations bilinear in x_i and the line coordinates

$$p_{ik} = x_i y_k - x_k y_i.$$

The x_i and p_{ik} satisfy four identities of the type

$$x_i p_{jk} + x_j p_{ki} + x_k p_{ij} = 0. \tag{3}$$

The equations (1) and (2) represent two quadrics through (y) which meet in a C_4 through (y). The lines of the cubic cone with vertex (y) through C_4 belong to a cubic complex. If we eliminate the x_i from (1) and (2) and any two of (3), we have the equation of the cubic complex each line (l) of which is mapped by a point (x) on (l).

From (1), (2), and (3) we see that the p_{ik} are quartic functions of x_i. Hence any linear complex meets the cubic complex in a cubic congruence which is mapped by a quartic surface $F_4(x)$. Two linear complexes meet the cubic complex in a cubic ruled

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surface which is mapped by the intersection of the two corresponding F_4's. Let

$$x_1=x_2=0, \qquad x_3=x_4=0$$

be the pair of polar lines common to the linear complexes. If the coordinates of a line through a point (x) meeting the polar lines are substituted in (1) and (2), we have two cubic surfaces through $x_1=x_2=0$, $x_3=x_4=0$ which meet in a residual C_7, the variable intersection of the two F_4's. The C_7 meets each polar line in 4 points and is of genus 4. The common curve of all the F_4's meets C_7 in 22 points and is of genus 8. Hence the cubic complex is mapped on S_3 by the linear system of quartic surfaces through a C_9, $p=8$.

3. *The Cubic Complex of the General Cubic Involutorial Transformation.* If (y) and (z) are conjugate points in the involutorial transformation they are polar conjugates with respect to three quadrics which by a suitable choice of coordinates have the equations*

$$\begin{aligned} &a_1x_1x_4 + a_2x_2x_4 + a_3x_3x_4 - x_2x_3 = 0,\\ (4)\qquad &b_1x_1x_4 + b_2x_2x_4 + b_3x_3x_4 - x_3x_1 = 0,\\ &c_1x_1x_4 + c_2x_2x_4 + c_3x_3x_4 - x_1x_2 = 0. \end{aligned}$$

Hence, if

$$g_{ij} = y_iz_j + y_jz_i, \qquad (i, j = 1, \cdots, 4),$$

the three bilinear equations which define the involution may be written in the form

$$(5)\qquad \begin{aligned} g_{23} = a_1g_{41} + a_2g_{42} + a_3g_{43},\ g_{31} = b_1g_{41} + b_2g_{42} + b_3g_{43},\\ g_{12} = c_1g_{41} + c_2g_{42} + c_3g_{43}. \end{aligned}$$

The g_{ij} and p_{ik} satisfy the identities which express the fact that $(g_{i1}, g_{i2}, g_{i3}, g_{i4})$, $(i=1, \cdots, 4)$, are on the line p_{ik}. Hence

$$(6)\quad g_{21}p_{34} + g_{23}p_{41} + g_{24}p_{13} = 0,\ g_{12}p_{43} + g_{14}p_{32} + g_{13}p_{24} = 0.$$

If we substitute in (6) for g_{23}, g_{31}, g_{12} from (5) and use x_i for g_{4i}, $(i=1, 2, 3, 4)$, then (6) is the form of (1) and (2) for this case. These two equations with any two of (3) define the cubic com-

* F. R. Sharpe and V. Snyder, *The* (1, 2) *correspondence associated with the cubic space involution of order two*, Transactions of this Society, vol. 25 (1923), pp. 1–12.

plex and map a line (l) of the complex on a point (x) of (l).* The cubic inversion† is the special case when

$$a_1 = b_2 = c_3 = 1,$$

and the other coefficients in (4) are zero.

4. *The First Type of the Quadratic Complex.* If the cubic complex of §2 reduces to a quadratic complex, then C_4 must break up into a cubic curve through (y) and either a line joining (y) to a fixed point O, or a fixed line (l).

In the first case the quadratic complex contains the bundle of lines through O, and the Kummer surface is a Steiner surface having three double lines through the triple point O. If the planes through the lines are

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0,$$

the quadratic complex has an equation of the form

$$a p_{23} p_{14} + b p_{31} p_{24} + c p_{12} p_{34} + F_2 = 0, \tag{7}$$

where F_2 is quadratic in p_{23}, p_{31}, p_{12}.

Consider the bilinear equation

$$p_{23}(Ax) + p_{31}(Bx) + p_{12}(Cx) = 0. \tag{8}$$

From equations (3), (7), and (8) we can derive the equations

$$\frac{p_{23}}{x_2(Cx) - x_3(Bx)} = \frac{p_{31}}{x_3(Ax) - x_1(Cx)} = \frac{p_{12}}{x_1(Bx) - x_2(Ax)},$$

which may be written

$$\frac{p_{23}}{f_1} = \frac{p_{31}}{f_2} = \frac{p_{12}}{f_3}, \tag{9}$$

where $f_i = 0$ is a quadric through a cubic C_3 which passes through O. Using (9) we find

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* For another method of mapping the cubic complex see D. Montesano, *Su di un complesso di rette del terzo grado*, Bologna Memoria, 1893, pp. 549–577. See p. 565.

† L. Godeaux, *Recherches sur les surfaces algébriques de genres zéro et de bigenre un*, Académie Royale de Belgique, Classe des Sciences, Bulletin, (5), vol. 12 (1926), pp. 892–904. See pp. 896–897.

$$(10)\qquad \begin{aligned} p_{23} &= f_1(ax_1f_1 + bx_2f_2 + cx_3f_3),\\ p_{31} &= f_2(ax_1f_1 + bx_2f_2 + cx_3f_3),\\ p_{12} &= f_3(ax_1f_1 + bx_2f_2 + cx_3f_3),\\ p_{14} &= x_1F_2(f) - (b-c)f_2f_3x_4,\\ p_{24} &= x_2F_2(f) - (c-a)f_3f_1x_4,\\ p_{34} &= x_3F_2(f) - (a-b)f_1f_2x_4. \end{aligned}$$

Hence if (x) is given, the p_{ik} are quintic functions of the x_i. Conversely, if the p_{ik} are given satisfying (7), we can write (8) in the form

$$d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = 0,$$

and, using (3), we can find for x_i the expressions

$$(11)\qquad x_i = d_1p_{i1} + d_2p_{i2} + d_3p_{i3} + d_4p_{i4}, \qquad (i = 1, 2, 3, 4),$$

which are quadratic in the p_{ik}. The quintic surfaces (10), $p_{ik}=0$, have C_3 for double curve. From (10) we see that if p_{23}, p_{31}, p_{12} are fixed, then the lines of the complex lie in the fixed plane

$$(12)\qquad x_1f_1 + x_2f_2 + x_3f_3 = 0,$$

and pass through the point (y) on the Steiner surface, where

$$(13)\qquad \begin{aligned} y_1 &= (b-c)f_2f_3, & y_2 &= (c-a)f_3f_1,\\ y_3 &= (a-b)f_1f_2, & y_4 &= F_2(f). \end{aligned}$$

The equations (13) map the Steiner surface on a plane (f_1, f_2, f_3). If we substitute the values of (y) from (13) in (8) we have the condition that the point (y) lies on the line in which (8) meets (12). This is a cubic relation in (f) so that on the plane (f) we have a cubic curve of genus 1 to which corresponds a C_6, $p=1$, on the Steiner surface meeting C_3 in the 9 points apart from O in which C_3 meets the Steiner surface. This C_6 lies on all the quintic surfaces $p_{ik}=0$. The intersection of the quadratic complex with a linear complex is therefore mapped by a quintic surface $F_5: C_3^2 C_6$. Two quintic surfaces meet in a variable C_7, $p=1$, meeting C_3 in 11 points and C_6 in 9 points. Three quintic surfaces meet in 4 variable points.

5. *The Cremona Transformation Associated with the Quadratic Complex.* Consider a second bilinear equation

(14) $p_{23}(A'x) + p_{31}(B'x) + p_{12}(C'x)$

$$= d_1' x_1 + d_2' x_2 + d_3' x_3 + d_4' x_4 = 0.$$

Solving as in (11) for x_i in terms of p_{ik} and substituting from (10), we have expressions which are linear in the $F_5: C_3^2 C_6$ with coefficients linear in (f). Hence we have a Cremona transformation of order 7 of the form $F_7: C_3^3 C_6$. A surface $F_7: C_3^3$ can be mapped on a plane by $C_5: 18A$, quintics through 18 fixed points. The curve C_3 is mapped by $C_9: 18A^2$ and the intersection of a variable $F_7: C_3^3$ by $C_8: 18A$. For the system $F_7: C_3^3 C_6$, the $C_8: 18A$ consist of $C_2: 3A$, image of the variable curve C_7, $p=0$, a fixed $C_3: 9A$, image of C_6, $p=1$, and a fixed curve C_9, $p=1$. The triple curve C_3 meets C_6 and C_9 in 9 and 15 points, respectively, and C_7 in 12 points, C_6 and C_9 meet in 9 points and meet C_7 in 6 points.*

6. *The Involutorial Transformation belonging to the Quadratic Complex.* If we replace (14) by

$$p_{23}H_1 + p_{31}H_2 + p_{12}H_3 = 0, \tag{15}$$

where $H_i = 0$ is a quadric, the surfaces $f_i = 0$ are cubics through a $C_7, p=5$, which passes through O, and the surfaces $p_{ik} = 0$ are of the form $F_7: C_7^2$. The line in which the plane (8) met the plane (12) is replaced by a conic. If the point (y) is on the conic, we have a relation of the fifth order in f_1, f_2, f_3 so that the surfaces F_7 are of the form $F_7: C_7^2 C_{10}$, $p=6$. The curve C_7 meets the Steiner surface in 25 points apart from O. Hence C_7 meets C_{10} in 25 points.† To a line of the complex correspond two points (x) on the line. Two surfaces $F_7: C_7^2 C_{10}$ meet in a variable C_{11}, $p=8$, meeting C_7 and C_{10} in 27 and 15 points, respectively. Three of the surfaces meet in 8 variable points. Given a plane $(kx)=0$, we can find the x_i in terms of the k_i and p_{ik} as in (11). Substituting these values for the (x_i) in (15) and for the p_{ik} from (10), we have a relation which is quadratic in the k_i of which (kx) is a factor. The other factor is the image of $(kx)=0$ in the involutorial transformation which interchanges the two points (x)

* Compare D. Montesano, *Su le trasformazioni univoche dello spazio che determinano complessi quadratice di rette*, Reale Istituto Lombardo Rendiconti, (2), vol. 25 (1892). See p. 803.

† Compare D. Montesano, Reale Istituto Lombardo Rendiconti, (2), vol. 25 (1892), p. 802.

on a line p_{ik} of the complex. This involutorial transformation is therefore of the form $F_{16}: C_7^5 C_{10}^2$. There are 25 trisecants of C_7 which meet C_{10} so that the surfaces all contain these 25 parasitic lines.

7. *The Second Type of the Quadratic Complex.* If the quadrics (7) and (8) are replaced by

$$ax_1 = bx_2, \; cx_1 = dx_2, \tag{16}$$

where a, b, c, and d are linear in the p_{ik}, the lines belong to the quadratic complex $ad-bc=0$. The intersection of this complex with a linear complex is mapped by a quartic surface $F_4:l^2$, where $l \equiv x_1 = x_2 = 0$. The intersection with a linear congruence is mapped by the intersection of two cubic surfaces through l and the directrices of the congruence and is therefore a variable C_6, $p=1$, meeting l in four points. Two of the surfaces $F_4:l^2$ meet therefore in a fixed C_6, $p=1$, meeting l in four points. Consider a second pair of equations

$$ax_3 = bx_4, \qquad cx_3 = dx_4, \tag{17}$$

which give a second mapping of $ad-bc=0$. The two mappings determine a (5, 5) Cremona transformation $F_5:l^3C_6$. There is therefore an additional simple basis curve C_5, $p=0$, meeting l and C_6 in four and eight points, respectively.*

8. *The Second Type of Involutorial Transformation.* If we replace (16) by

$$aH_1 = bH_2, \qquad cH_1 = dH_2, \tag{18}$$

where $H_1=0$, $H_2=0$ are quadrics meeting in C_4, $p=1$, the inter section of the quadratic complex $ad-bc=0$ with a linear complex is mapped by a sextic surface $F_6: C_4^2$, and with a lineaı congruence by the intersection of two quartic surfaces through the directrices of the congruence and through C_4, that is, by a C_{10}, $p=7$, meeting C_4 in 16 points. Two surfaces $F_6: C_4^2$ meet therefore in a variable C_{10}, $p=7$, and a fixed C_{10}, $p=7$, meeting C_4 in 16 points. On each line of the complex are two associated points (x) and (x'). Given a plane $\Sigma a_i x_i' = 0$, we find

* F. R. Sharpe, *Involutions of order n with an $(n-2)$-fold line*, Annals of Mathematics, (2), vol. 31 (1930), pp. 637–640.

$$x_1' = a_2 p_{12} + a_3 p_{13} + a_4 p_{14}$$

and three similar equations. From (18) we have

$$H_1' H_2 - H_2' H_1 = 0.$$

Substituting for the x_i' and for the p_{ik} in terms of the x_i we have a relation which is quadratic in a_i and has $\Sigma a_i x_i$ for a factor. The other factor is the image of $\Sigma a_i x_i' = 0$ by the involutorial transformation defined by (18). Hence the transformation is of the form $F_{13}: C_4^5 C_{10}^2$. There are 16 bisecants of C_4 which are bisecants of C_{10} and therefore lie on the F_{13}.*

9. *The Linear Complex.* Consider the linear complex $p_{12} = p_{34}$ and the bilinear equation

$$m_2 p_{13} + m_3 p_{14} + m_4 p_{23} + m_5 p_{24} + m_6 p_{34} = 0,$$

where the m_i are linear in x_i. Proceeding as in §2 we may show that the linear complex is mapped on S_3 by the linear system of cubic surfaces through a C_5, $p = 1$.

The transformations belonging to a linear or special linear complex have been discussed synthetically by Montesano and Pieri. If we use one equation bilinear in x_i and p_{ik} and either $p_{12} = p_{34}$ or $p_{12} = 0$, we can readily obtain the results of Montesano† and Pieri.‡

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* D. Montesano, Reale Istituto Lombardo Rendiconti, (2), vol. 25 (1892), p. 803.

† Compare D. Montesano, Napoli Accademia delle Scienze Fisiche e Matematiche, Rendiconti, (2), vol. 2 (1888), pp. 181–188. D. Montesano, Rendiconti dei Lincei, vol. 4 (1st semester), 1888, pp. 207–215, 277–285.

‡ Pieri, Rendiconti di Palermo, vol. 6 (1892), pp. 234–244.