

SETS OF LOCAL SEPARATING POINTS
OF A CONTINUUM*

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1. *Introduction.* Let M denote any locally compact metric continuum and let L be the set of all local separating points[†] of M . We proceed to establish the following six properties, of which, for our immediate purposes, the most useful is number (iv).

(i). *If U is any uncountable subset of L , there exists a point x of U which is a point of order 2 in M relative to U .*

This statement means that x is contained in arbitrarily small neighborhoods whose boundaries have in common with M just two points and these two points belong to U . A proof has already been given by the author (loc. cit.).

(ii). *If H is any connected subset of M , then $(\overline{H} - H) \cdot L$ is countable.*

For if not, (i) would give a point x of this set which could be separated in M from some point of H by two points not in H , which obviously is impossible since $H + x$ is connected.

(iii). *If H is any connected subset of M , the points of $H \cdot L$ which are not local separating points of H are countable.*

This results immediately from (i).

(iv). *If H is any connected subset of M such that $\overline{H} \subset L + C$, where C is some countable set, then H is a locally connected G_δ -set. Hence H is arcwise connected.*

By (ii) we see that $(\overline{H} - H) \cdot L$ and hence $\overline{H} - H$ itself is countable. Thus H is a G_δ -set. Now \overline{H} must be a regular curve, for by (i), all save a countable number of its points are points of order 2. Thus any connected subset of \overline{H} , and in particular H , is locally connected. That H is arcwise connected follows now by the well known theorem of Moore-Menger.[‡]

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[†] A point p is a local separating point of M provided some neighborhood V of p exists such that $M \cdot \overline{V} - p$ is separated between some pair of points belonging to the component of $M \cdot \overline{V}$ which contains p . See the author's paper in Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 305-314.

[‡] See R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13 (1932), p. 86; and K. Menger, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 193-218.

(v). If N is any continuum $\subset L + C$, where C is countable, then every connected subset of N is a locally connected G_δ -set and hence is arcwise connected.

This is a corollary to (iv).

(vi). If M is locally connected, then every connected subset H of L is the difference between an F_σ and a countable set.

For it is known* that in this case L is an F_σ , so that $M - L$ is a G_δ and hence so also is $(M - L) \cdot \overline{H} = (M - L) \cdot (\overline{H} - H)$. Whence, $H = [\overline{H} - (M - L) \cdot (\overline{H} - H)] - L \cdot (\overline{H} - H)$, and the first of these two sets is an F_σ and the second, by (ii), is countable.

2. THEOREM. In order that every connected subset of a continuum M be a G_δ it is necessary and sufficient that the set N of non-local-separating points of M be countable.

The sufficiency of the condition results immediately from (iv), in view of the fact that, for any connected subset H of M , we have $\overline{H} \subset M = L + N$, and N is countable.

To prove that the condition is necessary, we suppose, on the contrary, that N is uncountable. Let $M = M_1 + M_2$, where M_1 and M_2 are disjoint and totally imperfect† and where M_2 , say, contains uncountably many points of N . Then‡ $M_1 + L$ is connected and $M - (M_1 + L) = E$ is totally imperfect and uncountable. Thus E is not an F_σ and hence $M_1 + L$ is not a G_δ .

COROLLARY 1. If every connected subset of M is a G_δ , then M is a regular curve, no cyclic element of M has a continuum of condensation, and the end points of M are countable.

COROLLARY 2. If all save a countable number of the points of each cyclic element C of a locally connected continuum M are local separating points of C , then every connected subset of M is arcwise connected.

COROLLARY 3. If the non-local-separating points of each cyclic element of a locally connected continuum M are countable, then every connected subset of M will be a G_δ if and only if the end points of M are countable.

* See the author's paper, *Mathematische Annalen*, vol. 162 (1929), p. 318.

† That is, neither contains any perfect set. For a proof that such a division of M is possible, see F. Bernstein, *Leipziger Berichte*, vol. 60 (1908), p. 325 and Hausdorff, *Mengenlehre*, 1927, p. 156.

‡ See my paper *On the existence of totally imperfect sets . . .*, *American Journal of Mathematics*, vol. 55 (1933), pp. 146–152.

For any local separating point of a cyclic element of M is a local separating point of M and any non-local-separating point of M which is on a non-degenerate cyclic element of M must be an end point of M .

3. THEOREM. *If M is any locally compact continuum such that (a) no two maximal free arcs* in M abut and (b) $L \cdot \bar{R}$ is countable and $R \neq 0$, where R is the set of all ramification points (that is, points of order > 2) of M , then M contains a connected subset which is not arcwise connected.*

Proof. Let $M = M_1 + M_2$, where M_1 and M_2 are totally imperfect and disjoint. Set $E = M_1 + L$. Then since $M - E \subset M_2$, E is connected (loc. cit.). Furthermore $E \cdot \bar{R} = M_1 \cdot \bar{R} + L \cdot \bar{R}$, and since $L \cdot \bar{R}$ is countable it follows that $E \cdot \bar{R}$ is totally imperfect. Now if a and b are two points of E lying in different maximal free arcs† of M , there can exist no arc ab in E . For if ab is any arc in M from a to b , then $ab \cdot \bar{R}$ cannot be countable, (for if so, some two free arcs contained in ab would abut), and hence it must contain a perfect set. Thus ab cannot be $\subset E$, since $E \cdot \bar{R}$ is totally imperfect.

4. EXAMPLE. *There exists a regular curve C such that (a) no two free arcs of C abut, (b) $L \cdot \bar{R}$ is countable, and (c) \bar{R} is punctiform. Hence C contains a connected subset which is not arcwise connected.*

Let I be the unit interval and let K be the non-dense perfect set consisting of all numbers on I which can be expressed in the triadic number system using only the digits 0 and 2. Let I_1, I_2, \dots , be the segments on I complementary to K ordered in descending order of length and let p_1, p_2, \dots , be the end points of these segments, where p_{2n-1} and p_{2n} are the end points of I_n . Let $a_n = \sum_1^n 2^m$ and $P_n = \sum_1^n p_i$ and $P = \sum_1^\infty P_i$. Let us select in $K - P$ two sequences of points x_n and y_n such that

$$0 \leftarrow \dots < x_n < x_{n-1} < \dots < x_1 \\ < y_1 < \dots < y_{n-1} < y_n < \dots \rightarrow 1.$$

* An arc ab is free in M provided $ab - (a+b)$ is an open subset of M . Two such arcs are said to abut if they have a common end point.

† Clearly two such exist, otherwise M contains a punctiform connected set. See my paper, loc. cit.

In the upper half plane let us construct equilateral triangles with bases $x_2y_2, y_{4n-1}y_{4n+2}, x_{4n+2}x_{4n-1}, (n = 1, 2, 3, \dots)$, and in the lower half plane construct equilateral triangles with bases $y_{4n-3}y_{4n}$ and $x_{4n}x_{4n-3}, (n = 1, 2, 3, \dots)$. Let T_0 be the set of triangles so constructed.

Now the points (x_n) and (y_n) together with the points of P_1 divide I into a collection of intervals; let us omit I_1 from this collection and call J_1 the resulting collection. Now on each interval of J_1 let us construct a set of equilateral triangles exactly as we constructed T_0 on I . Let T_1 be the set of triangles obtained for all intervals of J_1 . Now the vertices of the triangles of T_1 that are on I together with the points of P_2 divide I into a set of intervals. Omit from this set the intervals I_1, I_2, I_3 and call J_2 the resulting collection. On each interval of J_2 construct a set of triangles as before and call T_2 the total set so constructed.

Continue this process indefinitely and let $C = I + \sum_0^\infty T_n$. Then clearly C is a continuum. The maximal free arcs in C are exactly the intervals I_n together with the two sides of each of the triangles of $[T_n]$ not on I , so that no two maximal free arcs in C abut. Furthermore the set R of ramification points of C is a subset of K such that $\bar{R} = K$. Thus \bar{R} is totally disconnected and C is a regular curve. Finally we note that if x is any point of $K - (P + R)$, then for each n there is a triangle t of T_n with base axb , where $0 < a < x < b < 1$; and since $\delta(t) < 1/(n+1)$ and x is not a ramification point, it follows that x is not a local separating point. Thus $L \cdot \bar{R} = L \cdot K \subset P + L \cdot R$, which shows that $L \cdot \bar{R}$ is countable, since both P and $L \cdot R$ are countable.

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