Since $a \leqq 374930473917097$, we have in each case $k \leqq 39111579$. Thus the problem of representing $N$ as the difference of squares was split into 8 parts. The first two parts were covered by the machine without any result. On the third run, however, the machine stopped almost at once at $x = 58088$. This gives

$$a = 556846584735, \quad b = 556644555032.$$

Hence we have the factorization

$$2^{79} - 1 = 2687 \cdot 202029703 \cdot 1113491139767.$$

It is not difficult to show that the factors are primes. This is the 13th composite Mersenne number to be completely factored. The author's recent report* on Mersenne numbers should be changed accordingly.

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# MATRICES WHOSE $s$TH COMPOUNDS ARE EQUAL

BY JOHN WILLIAMSON

If $A$ is a matrix of $m$ rows and $n$ columns and $s$ is any positive integer less than or equal to the smaller of $n$ and $m$, from $A$ can be formed a new matrix $A_s$ of ${}_mC_s$ rows and ${}_nC_s$ columns, the elements in the $t$th row of $A_s$ being the ${}_nC_s$ determinants of order $s$ that can be formed from the $t_1$th, $\cdots$, $t_s$th rows of $A$, and the elements in the $t$th column being the ${}_mC_s$ determinants of order $s$ that can be formed from the $t_1$th, $\cdots$, $t_s$th columns of $A$. The matrix $A_s$, so defined, is called the $s$th compound matrix of $A$. In the following note we discuss the necessary and sufficient conditions under which the $s$th compounds of two matrices are equal. We shall require the following lemmas.

LEMMA I. *The rank of the sth compound of a matrix $A$, whose rank is $r$, is ${}_rC_s$ if $r \geqq s$ and is zero if $s > r$.*†

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* This Bulletin, vol. 38 (1932), p. 384. Dr. N. G. W. H. Beeger has kindly called my attention to the fact that $2^{233}-1$ has two known prime factors and should be classified accordingly.

† Cullis, *Matrices and Determinoids*, vol. 1, p. 289.

LEMMA II. *The sth compound of the product of two matrices is the product of the sth compounds of the two matrices, or, in symbols,**

$$(AB)_s = A_s B_s. \tag{1}$$

THEOREM. *If $A$ is a matrix of rank $r$, the necessary and sufficient condition that $A_s = B_s$ is that*
(a) *the rank of $B$ be less than $s$ when $r < s$;*
(b) *there exist two non-singular matrices $C$ and $D$ such that*

$$CAD = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad CBD = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

*where $T$ and $S$ are two non-singular matrices of $r$ rows and columns such that $|T| = |S|$, when $r = s$;*
(c) *$A = \omega B$, where $\omega$ is an sth root of unity, when $r > s$.*

In case (a) if $A_s = B_s$, then $B_s = 0$ and by Lemma I the rank of $B$ is less than $r$. On the other hand if the rank of $B$ is less than $s$, then $B_s = 0 = A_s$. In case (b) the sufficiency of the condition follows from (1) and the fact that

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}_s = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}_s.$$

We now proceed to prove that the condition stated above is necessary. Since $A$ has rank $r$ there exist two non-singular matrices $C$ and $D$ such that

$$CAD = R = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix},$$

where $T$ is any non-singular $r$-rowed square matrix. If

$$CBD = F = \begin{pmatrix} S & G \\ H & K \end{pmatrix},$$

where $S$ is an $r$-rowed square matrix, $G$ an $r$ by $n-r$ matrix, $H$ an $m-r$ by $r$ matrix, and $K$ an $m-r$ by $n-r$ matrix, then, since $A_s = B_s$, it follows that $R_s = F_s$ and $|S| = |T| \neq 0$. Since $R_s$ contains only one element different from zero, every determinant of order $s$ that can be formed from $s-1$ columns of $S$ and one of $G$ is zero. If

$$S = (s_{ij}), \quad G = (g_{iq}), \quad (i, j = 1, 2, \cdots r; \quad q = 1, 2, \cdots, n-r),$$

* H. W. Turnbull, *Determinants, Matrices and Invariants*, pp. 81–82.

and $S_{ij}$ is the cofactor of $s_{ij}$ in $S$, then

$$(2) \qquad \sum_{i=1}^{r} S_{ij} g_{iq} = 0, \quad (j = 1, 2, \cdots, r; q = 1, 2, \cdots, n - r).$$

For a fixed $q$, the equation (2) represents a set of $r$ homogeneous equations in the $r$ unknowns $g_{iq}$, and since $|S_{ij}| = |S|^{r-1} \neq 0$, it follows that $g_{iq} = 0$. Accordingly $G = 0$ and by a similar argument $H = 0$, so that $F$ has the form

$$\begin{pmatrix} S & 0 \\ 0 & K \end{pmatrix}.$$

But, since $S$ is non-singular, at least one of the quantities $S_{ij} \neq 0$. If $k$ is any element of $K$, we observe that $kS_{ij}$ is an element of $F_s$ which must be zero, and therefore $K = 0$.

In case (c), the sufficiency of the condition is an immediate consequence of (1). If the rank $r$ of $A$ is greater than $s$, there must exist in $A$ a submatrix $T$ of $s+1$ rows and columns, which is non-singular. Without any loss of generality we may suppose that

$$A = \begin{pmatrix} T & K \\ L & M \end{pmatrix}, \quad B = \begin{pmatrix} S & H \\ P & Q \end{pmatrix},$$

where $S$ is an $(s+1)$-rowed square matrix. From $A_s = B_s$, we deduce that $T_s = S_s$ and

$$|T_s| = |T|^s = |S_s| = |S|^s,$$

so that

$$(3) \qquad |S| = \omega |T|,$$

where $\omega$ is an $s$th root of unity. Moreover*

$$(T_s)_s = |T|^{s-1}T = (S_s)_s = |S|^{s-1}S,$$

so that, by (3), $S = \omega T$. Since $T$ is non-singular, there must exist in $T$ a non-singular submatrix $T'$ of $s$ rows and columns. If $A'$ denote a matrix obtained from $A$ by a rearrangement of rows and columns, so that $T'$ occurs in the top left-hand corner of $A'$, and $B'$ is the matrix obtained from $B$ by exactly the same rearrangement, then

* $(T_s)_s$ denotes the $s$th compound of $T_s$. That $(T_s)_s = |T|^{s-1}T$ is simply the well known theorem on the adjugate of the adjugate of a matrix.

$$A' = \begin{pmatrix} T' & K' \\ L' & M' \end{pmatrix}, \quad B' = \begin{pmatrix} \omega T' & H' \\ P' & Q' \end{pmatrix},$$

and from $A_s = B_s$ it follows that $A_s' = B_s'$. If

$$T' = (t_{ij}), \quad K' = (k_{iq}), \quad H' = (h_{iq}),$$
$$(i, j = 1, 2, \cdots, s; \; q = 1, 2, \cdots, n - s),$$

and $T_{ij}$ denote the cofactor of $t_{ij}$ in $T'$, then

$$\sum_{i=1}^{s} T_{ij} k_{iq} = \sum_{i=1}^{s} \omega^{s-1} T_{ij} h_{iq}, \qquad \text{or} \qquad \sum_{i=1}^{s} T_{ij}(k_{iq} - \omega^{s-1} h_{iq}) = 0.$$

But, since $|T_{ij}| \neq 0$, $k_{iq} - \omega^{s-1} h_{iq} = 0$ or $H' = \omega K'$. Similarly it may be shown that $P' = \omega L'$. Let $T''$ be a submatrix of $T'$ of order $s-1$ which is non-singular. If $m_{ij}$ is any element of $M'$ and $q_{ij}$ the corresponding element of $Q'$, the determinant of order $s$ formed from $A'$ of the $s-1$ rows and columns of which $T''$ is composed and the row and column in which $m_{ij}$ lies is equal to the corresponding determinant formed from $B'$. But from the equality of these two determinants it follows that $m_{ij}|T''| = \omega^{s-1} q_{ij} |T''|$ and therefore, since $|T''| \neq 0$, it follows that $Q' = \omega M'$, $A' = \omega B'$, and $A = \omega B$. This completes the proof of the theorem.

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# REMARKS ON PROPOSITIONS $*1\cdot 1$ AND $*3\cdot 35$ OF PRINCIPIA MATHEMATICA†

BY B. A. BERNSTEIN

1. *Object.* Among the propositions of the theory of deduction underlying Whitehead and Russell's *Principia Mathematica* are the two following:

$*1\cdot 1$. *Anything implied by a true elementary proposition is true.*

$*3\cdot 35$. $\vdash : p \cdot p \supset q \cdot \supset \cdot q$.

The authors interpret $*3\cdot 35$ as "if $p$ is true, and $q$ follows from it, then $q$ is true," and they remark that $*3\cdot 35$ "differs

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† Presented to the Society, September 2, 1932.