$$A' = \begin{pmatrix} T' & K' \\ L' & M' \end{pmatrix}, \quad B' = \begin{pmatrix} \omega T' & H' \\ P' & Q' \end{pmatrix},$$

and from $A_s = B_s$ it follows that $A_s' = B_s'$. If

$$T' = (t_{ij}), \quad K' = (k_{iq}), \quad H' = (h_{iq}),$$
$$(i, j = 1, 2, \cdots, s;\ q = 1, 2, \cdots, n - s),$$

and T_{ij} denote the cofactor of t_{ij} in T', then

$$\sum_{i=1}^{s} T_{ij} k_{iq} = \sum_{i=1}^{s} \omega^{s-1} T_{ij} h_{iq}, \qquad \text{or} \qquad \sum_{i=1}^{s} T_{ij}(k_{iq} - \omega^{s-1} h_{iq}) = 0.$$

But, since $|T_{ij}| \neq 0$, $k_{iq} - \omega^{s-1} h_{iq} = 0$ or $H' = \omega K'$. Similarly it may be shown that $P' = \omega L'$. Let T'' be a submatrix of T' of order $s-1$ which is non-singular. If m_{ij} is any element of M' and q_{ij} the corresponding element of Q', the determinant of order s formed from A' of the $s-1$ rows and columns of which T'' is composed and the row and column in which m_{ij} lies is equal to the corresponding determinant formed from B'. But from the equality of these two determinants it follows that $m_{ij}|T''| = \omega^{s-1} q_{ij}|T''|$ and therefore, since $|T''| \neq 0$, it follows that $Q' = \omega M'$, $A' = \omega B'$, and $A = \omega B$. This completes the proof of the theorem.

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REMARKS ON PROPOSITIONS $*1\cdot 1$ AND $*3\cdot 35$ OF PRINCIPIA MATHEMATICA†

BY B. A. BERNSTEIN

1. *Object.* Among the propositions of the theory of deduction underlying Whitehead and Russell's *Principia Mathematica* are the two following:

$*1\cdot 1$. *Anything implied by a true elementary proposition is true.*

$*3\cdot 35$. $\vdash : p \cdot p \supset q \cdot \supset \cdot q$.

The authors interpret $*3\cdot 35$ as "if p is true, and q follows from it, then q is true," and they remark that $*3\cdot 35$ "differs

† Presented to the Society, September 2, 1932.

from $*1\cdot 1$ by the fact that it does not apply only when p *really* is true, but requires merely the *hypothesis* that p is true."† It is my object to make a few remarks on these interpretations of $*1\cdot 1$ and $*3\cdot 35$.

2. *On Proposition* $*3\cdot 35$. With regard to $*3\cdot 35$, the *Principia's interpretation is inadmissible.* For in this interpretation the authors read p, $p\cdot q$, $p \supset q$, respectively, as "p is true," "p is true and q is true," "p implies q"; and *each of these readings violates the distinction between p and $\vdash\cdot p$* properly made *earlier in the theory of deduction* (see the *Principia*, vol. I, pp. 91–92). Or, to state the matter in another way, the symbols p, $p\cdot q$, $p \supset q$ are each *elementary* propositions, and hence *cannot* be read, as the *Principia* reads them, as propositions *about* elementary propositions. The correct readings of p, $p\cdot q$, $p\supset q$ are, respectively: "p," "p and q," "not-p or q." The correct reading of $*3\cdot 35$ is, then:

"The proposition 'not-[p and (not-p or q)] or q' is true."‡

3. *On Proposition* $*1\cdot 1$. With regard to $*1\cdot 1$, I hold that *the Principia's view that* $*1\cdot 1$ *can "apply only when p really is true," is not justified.* Neither the mere wording of $*1\cdot 1$ nor the use of $*1\cdot 1$ in the theory of deduction justifies the view that in $*1\cdot 1$ "p *really* is true." The mere wording of $*1\cdot 1$ seems to me to say the same thing as the proposition X following:

X. If p is true, and p implies q, then q is true.

And *this proposition is true even if p is false.*

As to the *use* of $*1\cdot 1$ in the theory of deduction, I have pointed out elsewhere§ that though the authors say "we cannot express this principle symbolically," they employ $*1\cdot 1$ as if it were written in the form Y following:

Y. If $\vdash\cdot p$ and $\vdash\cdot p\supset q$, then $\vdash\cdot q$.||

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† See the *Principia*, p. 110. (Here and in later footnotes the *Principia* referred to is vol. I, 2d ed.)

‡ Compare the remarks on $\vdash\cdot p$ and $p\supset q$ in my review of the revised edition of the *Principia*, this Bulletin, vol. 32 (1926), pp. 711–713. Compare also the remarks on $p\supset q$ and $p\equiv q$ in my article *On proposition* *4.78 *of Principia Mathematica*, this Bulletin, vol. 38 (1932), pp. 388–391.

§ In my *Whitehead and Russell's theory of deduction as a mathematical science*, this Bulletin, vol. 37 (1931), pp. 480–488.

|| See, for example, the note following the proof of *2.15 in the *Principia.* See also p. xviii, regarding Nicod's "rule of inference."

But this proposition Y is simply proposition X with the primitive ideas of the theory of deduction symbolized.† Hence, the *use of* $*1\cdot 1$, as well as the wording of $*1\cdot 1$, fails to support the view that in $*1\cdot 1$ "p really is true."

Observe that X is precisely the proposition which the authors of the *Principia* take as the interpretation of $*3\cdot 35$. We thus find that the *Principia's* interpretation of $*3\cdot 35$, an interpretation which the authors hold to be inapplicable to $*1\cdot 1$, *really fits* $*1\cdot 1$ *and does not fit* $*3\cdot 35$.

4. *On* $*1\cdot 1$ *and* $1\cdot 1$. It should be noted that proposition $*1\cdot 1$ is not the same as proposition $1\cdot 1$, the proposition that I used previously‡ as the Boolean form of $*1\cdot 1$. Proposition $1\cdot 1$ is:

$1\cdot 1$. There exists a K-element 1 such that from $p=1$ and $p'+q=1$ follows $q=1$.

This proposition is not a mere Boolean symbolization of $*1\cdot 1$. Such a symbolization would be simply:

Z. From $p=1$ and $p'+q=1$ follows $q=1$.

That is, the mere Boolean symbolization of $*1\cdot 1$ would be simply proposition $1\cdot 1$ with the omission of the clause stating that the element 1 exists. This clause, however, as I explained in my paper (loc. cit.) dealing with the Boolean translation of the theory of deduction, is involved in the *Principia's* $*1\cdot 1$. My proposition $1\cdot 1$ is thus neither $*1\cdot 1$ nor $*3\cdot 35$; but it is $*1\cdot 1$ in which is made explicit the fact, implied in $*1\cdot 1$, that there exists a "true" proposition 1.§

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† Note that $\vdash\cdot p\supset q$ may properly be read "p implies q." (For a discussion of the symbolization of "p implies q" see my review of the *Principia*, loc. cit., and my article on proposition *4.78, loc. cit.) Note also that the unsymbolized "if . . . then" and "and" are properly *outside* the theory of deduction. (For a discussion of the ideas *within* a mathematical science and the ideas *outside* the science see my *Whitehead and Russell's theory of deduction as a mathematical science*, loc. cit.)

‡ In my *Whitehead and Russell's theory of deduction as a mathematical science*, loc. cit.

§ Paul Henle (this Bulletin, vol. 38 (1932), p. 409) says of my proposition $1\cdot 1$ that it "is not an accurate transcription of $*1\cdot 1$ unless the convention be adopted that the postulate is not satisfied by any case in which the *hypothesis* is not satisfied," and he merely refers to p. 110 of the *Principia*, the page containing the interpretation of $*3\cdot 35$ and the remark concerning the relation of $*3\cdot 35$ to $*1\cdot 1$ quoted in §1 above. It seems to me that the question of

5. *On* $*1\cdot 1$ *and* $*1\cdot 11$. It should be noted further that, unless the authors use their $*1\cdot 1$ in some such form as Y above, they will not have provided proofs for a host of theorems in Section A of the *Principia*. For they use $*1\cdot 11$ in the proofs of many theorems; but, while they tell us in the second edition of the *Principia* to drop $*1\cdot 11$ from the list of primitive propositions, *they seemingly fail to tell us explicitly what to substitute for* $*1\cdot 11$ *in the proofs employing it.* Proposition Y above seems to me well suited as the desired substitute. As an illustration of the working of Y in place of $*1\cdot 11$, see my derivation† of Nicod's postulates from the primitives of the *Principia*. I may add that that paper summarizes in a way my position with regard to the mathematics of Section A of the *Principia*.‡

the accuracy of $1\cdot 1$ as a translation of $*1\cdot 1$ should not be decided by "convention," unless the language of $1\cdot 1$ is ambiguous. But it is clear to me that my $1\cdot 1$ is not ambiguous, and *is* "satisfied by any case in which the hypothesis is not satisfied." Nevertheless, I hold that $1\cdot 1$ is an accurate translation of $*1\cdot 1$, as I have tried to show in the above discussions of $*1\cdot 1$ and $1\cdot 1$.

† Bernstein, *On Nicod's reduction in the number of primitives of logic*, Proceedings of the Cambridge Philosophical Society, vol. 28 (1932), pp. 427–432.

‡ Since the above was written, an admirable paper by Huntington has appeared (Transactions of this Society, vol. 35 (January, 1933), pp. 274–304), in which a position is taken regarding the *Principia* somewhat different from mine. In Appendix II, Huntington presents a "set of postulates from which all the propositions, both 'formal' and 'informal,' in Section A of the *Principia*, are deducible." The postulates $1\cdot 1$–$1\cdot 71$ which form my version of Section A, however, I based on the *formal* propositions in Section A, in accordance with the *Principia's* statement (p. vii) that "our logical system is wholly contained in the numbered propositions, which are independent of the Introduction and the Summaries. The Introduction and the Summaries are wholly explanatory, and form no part of the chain of deductions." With regard to Huntington's observation (p. 291) that my *Principia* postulate $1\cdot 5$ is derivable from my other postulates (in accordance with Bernays' findings in examining the *Principia* primitives themselves), I may say that in my paper in the June, 1931, issue of this Bulletin, the chief aim was to express the *Principia* primitives in customary mathematical language; that the redundancies removed there are only *obvious* redundancies discovered in the pursuance of the main aim of the paper; and that at the time of writing the paper, I was unaware of Bernays' work.

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