SOME INVOLUTORIAL LINE TRANSFORMATIONS*

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1. *Introduction.* In a paper published in this Bulletin,† the author discussed three involutorial line transformations obtained by considering the transversals of one line each of four plane pencils; one line each of two plane pencils and two generators of a quadratic regulus; and two generators each of two quadratic reguli. In this paper we shall discuss those involutorial transformations between the two transversals of three generators of a cubic regulus and one line of a plane pencil; and the transversals of four generators of a quartic regulus. We shall also discuss an involutorial line transformation defined by two plane harmonic homologies. The methods used are algebraic and the results obtained are interpreted as point transformations on a certain quadratic hypersurface in S_5.

2. *Cubic Regulus and Plane Pencil.* Consider the ruled surface

$$F_3 \equiv z_1^2 z_3 - z_2^2 z_4 = 0,$$

and the pencil of lines in the plane

$$\alpha \equiv z_1 + z_2 - z_3 + z_4 = 0,$$

with vertex $A \equiv (1, 0, 1, 0)$. The Plücker coordinates of a generator of F_3 and those of a line of the pencil (A, α) are $(0, -k^2, 1, 0, k, k^3)$ and $(1, 0, -m, m, 1+m, 1)$, respectively, where the k and m are parameters. A general line (y) with coordinates y_i, $(i=1, \cdots, 6)$, meets three generators of F_3 and one line of (A, α). These four lines, in general mutually skew, have a second transversal (x) whose co-ordinates are found to be

$$x_i = \phi_i(y), \qquad (i = 1, \cdots, 6), \tag{1}$$

where the $\phi_i(y)$ are cubic functions in y_j. Thus the transformation is of third order.

The invariant locus of the transformation is a quadratic complex whose equation is found in the manner described in the pre-

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vious article by the author.* The singular elements of (1) are the two directrix lines of F_3, the bundle of lines (A), and the plane field of lines (α).

To the double directrix d_2 of F_3 correspond all of the lines of the special linear complex with axis a_{d_2}, the line of (A, α) met by d_2, and to the single directrix d_1 correspond the lines of the special linear complex whose axis is a_{d_1}. To each line of the bundle (A) corresponds the quadratic regulus determined by the three generators of F_3 met by the line of (A). Likewise, to each line of the plane field (α) corresponds the quadratic regulus determined by the three generators of F_3 met by the line of (α). The lines of (A) and of (α) belong also to the complex of invariant lines.

On the quadratic hypersurface V_2 in S_5 the point transformation (1) has for singular elements the points which represent the directrices d_1, d_2 of F_3 and the ρ-planes and ω-planes which represent the plane field (α) and the bundle (A), respectively.

To a general line on V_2 corresponds a cubic curve with the line as a bisecant. This can be seen from the fact that a line on V_2 represents a plane pencil in S_3 and a cubic curve on V_2 represents a cubic regulus in S_3. The two points of intersection represent the two invariant lines belonging to the pencil.

To a general plane of points on V_2 corresponds a cubic surface on V_2, for, in S_3, the image in (1) of a general bundle of lines is a congruence of order three, composed of the ∞^1 cubic reguli which are images of the ∞^1 pencils having their vertices at the vertex of the bundle; by duality, the image of a plane field of lines is a congruence of class three.

To the intersection of V_2 with a tangent hyperplane corresponds a three-dimensional locus of order three, for in S_3 the conjugate of a linear complex in (1) is a cubic complex, and the representation on V_2 of a special linear complex of S_3 is the intersection with V_2 of a hyperplane tangent to V_2 at the point which represents the axis of the complex.

3. *Quartic Regulus.* Let us now consider the involutorial line transformation determined by the transversals of four generators of a quartic regulus. Those ruled quartic surfaces having a straight line directrix are not considered, for this directrix is the

* Loc. cit.

second transversal of the four generators met by any arbitrary line of S_3. Hence only those quartics having a twisted cubic for a double curve and no simple directrix line are considered.

Let the double cubic be represented parametrically by

$$C_3 \equiv z_1 = t^3;\ z_2 = t^2;\ z_3 = t;\ z_4 = 1.$$

If a line join a point t of C_3 to a point s of C_3, its Plücker coordinates are

$$[1, -(t+s), ts, t^2s^2, ts(t-s), ts(t^2-s^2), (t^3-s^3)].$$

Now, a non-symmetric linear relation between t and s makes these coordinates of fourth degree in t, so that the ruled surface is a quartic. For the special case where $t=s$, the surface is the developable of C_3, and where $t+s=0$ we obtain the quadratic regulus common to three special linear complexes.

By choosing any non-symmetric relation between t and s, the transformation is found to be linear. It is the polarity as to the linear complex to which the given surface belongs. On V_2 in S_5, then, we have a projective transformation defined.

4. *The Harmonic Homologies.* Consider now in the plane $\alpha \equiv z_1 = 0$, the harmonic homology having its center at $A \equiv (0, 0, 0, 1)$ and as axis the line $a \equiv z_1 = 0$, $z_3 - z_4 = 0$. In the plane $\beta \equiv z_4 = 0$, consider the harmonic homology with center $B \equiv (1, 0, 0, 0)$ and axis $b \equiv z_4 = 0$, $z_2 - z_1 = 0$. A general line (y) in space meets α in a single point P and β in a point Q. In the homology $[a]$ in α, we have $P \sim P'$, and in the homology $[b]$ in β, we have $Q \sim Q'$. The line (y) then has for conjugate in the transformation the line $P'Q'$.

The equations of $[a]$ are

$$\rho z_1' = 0;\ \rho z_2' = z_2;\ \rho z_3' = z_3;\ \rho z_4' = 2z_3 - z_4,$$

while the equations of $[b]$ are

$$\rho z_1' = 2z_2 - z_1;\ \rho z_2' = z_2;\ \rho z_3' = z_3;\ \rho z_4' = 0.$$

Now an arbitrary line (y) with Plücker coordinates $(y_1, y_2, y_3, y_4, y_5, y_6)$ meets $z_1 = 0$ in the point $(0, y_4, y_5, y_6)$ and $z_4 = 0$ in $(y_6, -y_2, y_1, 0)$.

By $[a]$, the point $P \equiv (0, y_4, y_5, y_6) \sim P' \equiv (0, y_4, y_5, 2y_5 - y_6)$;

and, by $[b]$, the point $Q\equiv(y_6, -y_2, y_1, 0)\sim Q'\equiv(-y_6-2y_2, -y_2, y_1, 0)$. The Plücker coordinates of the line $P'Q'$ are

$$(2)\qquad \begin{cases} x_1 = y_1(2y_5 - y_6), \; x_2 = y_2(2y_5 - y_6), \; x_3 = y_3y_6, \\ x_4 = y_4(-2y_2 - y_6), \; x_5 = y_5(-2y_2 - y_6), \\ x_6 = (2y_5 - y_6)(-2y_2 - y_6). \end{cases}$$

Then (2) are the equations of transformation.

The line AB is invariant. Its coordinates are $(0, 0, 0, 0, 0, 1)$, which, when substituted in (2), make $x_6\neq 0$ and $x_i=0$, $(i=1, \cdots, 5)$. Likewise, the pencils of lines having their vertices at A, B and lying in the planes A, b, and B, a, respectively, are such that each line is invariant, as can be determined by making the substitutions in (2). There is, in addition, a bilinear congruence of lines, namely, all lines meeting a, b, which are invariant. The equations of the congruence are $x_5-x_6=x_2+x_6=0$.

Consider a line t_α in the plane α. It meets β in a point C of the line $c\equiv\alpha\beta$. In β there is a fixed point C_β, image by $[b]$ of C. In α, t_α has an image t_α' by $[a]$. Hence, since any point of t_α can be considered as P, the conjugate of t_α is the entire pencil (C_β, t_α'). Thus t_α is singular. As t_α revolves about C, t_α' describes a pencil whose vertex is C_α, the image by $[a]$ of C. Hence the conjugates of the lines of the pencil (C, α) belong to the bundle (C_β), all of whose pencils which are conjugates of t_α have their planes through $C_\alpha C_\beta$. As C describes the line c, C_β describes the line c_β in β. Thus the conjugate of the plane field of lines (α) is the special linear complex whose axis is c_β.

In like manner, the conjugate of the plane field of lines (β) is the special linear complex whose axis is c_α.

The intersection of the complexes $|c_\alpha|$, $|c_\beta|$ is a bilinear congruence which is the conjugate of the line c. Its equations are easily seen from (2) to be $2x_5-x_6=2x_2+x_6=0$.

Any line through a point C on c has for its conjugate the line $C_\alpha C_\beta$. Let the coordinates of C be $(0, 1, \lambda, 0)$ where λ is a parameter. The coordinates of C_α are $(0, 1, \lambda, 2\lambda)$ and of C_β $(-2, 1, \lambda, 0)$. Hence the coordinates of the line $C_\alpha C_\beta$ are quadratic in λ, so that to each line of a quadratic regulus $\{c\}_2$ corresponds by (2) a whole bundle of lines whose vertex lies on c. The regulus $\{c\}^2$ is singular.

If t_α revolve about a point $T\equiv(0, a, b, c)$, then t_α' describes the pencil (T', α), where $T' \equiv (0, a, b, 2b-c)$, and C_β describes c_β. We have seen that t_α is singular and has as conjugate the pencil whose vertex is C_β and whose plane is (C_β, t_α'). Thus, the pencil (T, α) has as conjugate a congruence composed of the pencils whose vertices are on c_β and whose planes pass through T'. Let λ be a parameter. Then one representation of (T, α) will be the lines t_α joining $(0, a, b, c)$ to $(0, 0, 1, \lambda)$. Then t_α' will be the lines joining T' to $(0, 0, 1, 2-\lambda)$. The point C, where t_α meets c, is $(0, a\lambda, b\lambda-c, 0)$, and hence C_β is $(-2a\lambda, a\lambda, b\lambda-c, 0)$. The lines through an arbitrary point M and C_β have coordinates which are linear in λ. If any of these lines meet the corresponding t_α' they will belong to the congruence. Putting on the condition that two lines intersect, we obtain a quadratic equation in λ. Hence the order of the congruence is 2. An arbitrary plane meets c_β in one point C_β and meets the corresponding t_α' in one point. Hence the class of the congruence is 1.

On V_2 in S_5, a plane pencil of S_3 is represented by a line. Hence the conjugate by (2) of any line in the ρ-plane which is the representation of (α) is a ruled surface R on V_2 with a straight line directrix. The directrix is the representation of the pencil of S_3 whose vertex is T' and whose plane is (T', c_β). Every ω-plane of V_2 meets R in two points and every ρ-plane meets it in one. The S_3 determined by these two intersecting planes therefore meets R in three points; thus R is of order 3.

Consider the conjugate of the lines of an arbitrary plane pencil (T, τ). Each line of the pencil meets α and β in one point each, and hence has a definite conjugate. Since the conjugate of the pencil is the ∞^1 lines joining the corresponding points of two projective ranges on two skew lines, it is a quadratic regulus.

On V_2 in S_5, the conjugate of an arbitrary straight line (representation of a pencil in S_3) is a conic (representation of a quadratic regulus). This follows directly from (2) considered as a point transformation on V_2.

Consider now a plane field of lines in S_3. The plane μ of the field meets α and β in a line each, and the conjugate of (μ) is a $(1, 1)$ congruence whose directrices are the images by $[a]$, $[b]$ of (μ, α), (μ, β), respectively. On V_2 in S_5 a general ρ-plane is transformed into a quadric surface.

The lines of S_3 belonging to a bundle (M), however, are transformed by (2) into a congruence (3, 1). The fields (α), (β), projected from P upon, say, α are such that when P is joined to the self-corresponding points, then the line PA_i, ($i=1, 2, 3$), passes through B_i, the point associated with A_i in the projectivity. Hence the congruence is of order three. A general plane π meets c_α, c_β in one point each, and the line joining π, c_α to π, c_β is the only line of the congruence lying in π. The class is therefore one.

On V_2 of S_5, a general ω-plane is transformed into a quartic surface.

The lines of S_3 belonging to a bilinear congruence with axes d_1, d_2 are transformed by (2) into a (4, 2) congruence. To see this, let us consider the quadratic regulus determined by d_1, d_2, c_β. These generators meet α in a conic α^2, whose image by $[a]$ is another conic α'^2, and meet β in a conic β^2, whose image by $[b]$ is a conic β'^2. The parameter λ of a point on α'^2 enters to the second degree; since the points of β'^2 are projective with those of α'^2, λ enters to degree two in defining the corresponding point on β'^2. The coordinates of the line joining an arbitrary point N to a point of α'^2 are of fourth degree in λ, when the line also passes through the corresponding point of β'^2. Thus there are four lines of the conjugate congruence which pass through N. An arbitrary plane ν meets α'^2 and β'^2 in two points each, and the lines joining the corresponding points are two in number. Hence the congruence is of class 2.

On V_2 of S_5, the conjugate of a quadric point cone (intersection with V_2 of the S_3 common to two tangent hyperplanes) is a surface of order six, for a general ρ-plane meets it in two points while a general ω-plane meets it in 4.

The conjugate of a special linear complex in S_3 is a quadratic complex, as is seen by the equations (2) which are of second degree. On V_2 of S_5, the intersection of a tangent hyperplane has for conjugate a three-dimensional locus of order two.

The author is now considering the case where there is in α, β, or both, some other Cremona involution.

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