THE MATHEMATICAL WORK OF OLIVER DIMON KELLOGG

Oliver Dimon Kellogg was born on July 10, 1878, at Linnwood, Pennsylvania. His untimely death on August 26, 1932, from an unlooked-for heart attack while climbing, came as a heavy blow to all those who were privileged to know him as colleague and friend. His quick, generous nature and unusual charm of personality were united with a versatile and original mind. The full story of Kellogg's many successful efforts to help others would be an extraordinary one, only to be guessed at by those who knew him intimately; and, in order to judge his mathematical achievements, it is necessary not only to consider his published work but to take into account his modesty and his readiness to share nascent ideas with others.

His interest in mathematics was aroused as an undergraduate at Princeton, largely through contact with H. B. Fine and E. O. Lovett. After securing an A.B. degree in 1899 with high honors in mathematics, he stayed on for a year of graduate study as J. S. K. Fellow and obtained an M.A. degree in 1900. His appointment was then extended for a second year, which he spent at the University of Berlin.

At the end of this period Kellogg, eager to engage in serious research, went to Göttingen. Within a few months, Fredholm's preliminary communication *Sur une nouvelle méthode pour la résolution du problème de Dirichlet,* before the Swedish Academy of Sciences, had made known the explicit solution of a large class of linear integral equations. This discovery seemed to promise an answer to all the outstanding linear problems in analysis, such as the Dirichlet problem and the Riemann problem of monodromic groups. However, Fredholm's intention of taking up such applications was never realized because of the extraordinary rapidity and skill with which Hilbert and his students plunged into the field. Thus Kellogg, inspired by the lectures of Hilbert, began to occupy himself with some of the problems suggested, in particular with the Dirichlet problem for plane regions bounded by a finite number of regular curves meeting at "corners," when Fredholm's solution was no longer directly available.

Kellogg's first paper was a note (1)* in the Göttinger Nachrichten of 1902 in which he provided a direct proof of Fredholm's fundamental inversion formula. His interesting doctoral thesis of the same year, entitled *Zur Theorie der Integralgleichungen und des Dirichlet'schen Prinzips* (2), was written in close connection with Hilbert's lectures and was thought of by Kellogg as containing only a "kleinen Beitrag." After receiving his Ph.D. degree in January, 1903, Kellogg remained in Germany until the fall; in the two following years he was instructor in mathematics at Princeton. During this period he wrote two short papers (4), (5) which were published in the Mathematische Annalen. The first of these was in part a "Neubearbeitung" of his dissertation, while the second attempted to solve the monodromic group problem of Riemann along

* See (1) of the bibliographical list below. Numbered references are to that list.

the lines indicated by Hilbert in his lectures of 1901–1902 and published in the Göttinger Nachrichten of 1905.

It is natural to consider the thesis and these two papers together since they were the immediate outcome of his studies at Göttingen. The chief contribution of the thesis and these two papers lies in the more extended treatment of certain singular integral equations in which the Cauchy principal value can be employed; this type of equation arises in connection with the solution of the Dirichlet problem in the plane, looked at as the potential of a simple distribution on the boundary. It is interesting to remark that Poincaré in his second Göttingen Lecture of 1909 pointed out how Kellogg's methods might be replaced by others involving complex integration.

Kellogg himself soon came to regard this work as definitely unsatisfactory and never refers to it in his later papers, despite the fact that so much of his research dealt with potential theory. The reasons for this feeling were probably the following: In the first place he had not solved the original problem proposed by Hilbert. This was no fault of his, for the problem of attacking the Dirichlet problem for regions with "corners" by a direct extension of the Fredholm methods remains unsolved to this day; thus in his 1926 address, *Recent progress with the Dirichlet problem* (27), before the Society, Kellogg said: "The method of integral equations, developed by Hilbert, because of its powerful character, gave great impetus to the study of the Dirichlet problem. . . . But it was hindered in aiding substantially the progress of the Dirichlet problem by its use of the double distribution, which carried with it the demand for a fairly smooth boundary." In the second place the adjacent problems were followed up by Hilbert also, so that it was almost impossible to dissociate his own work from the larger achievements of Hilbert. Thirdly, Kellogg soon saw that what he had written was incomplete and not always correct. In this connection it is interesting to remark that certain of these oversights were repeated in subsequent work of Poincaré and Hilbert.*

The scientific importance to Kellogg of his Göttingen sojourn can scarcely be overstated. He had witnessed at first hand the development of important mathematical ideas by a great master, and had taken a definite part in the application of these ideas to potential theory; and his research experience had led him to see for himself that a more solid foundation was absolutely necessary. In consequence he turned his attention to a thoroughgoing study of the field of potential theory, to which so much of his energy was to be devoted.

In 1905 Kellogg was called to the University of Missouri, where the scientific environment proved happy and stimulating despite a considerable amount of teaching and administrative duties. The first results of this period were two important articles, *Potential functions on the boundary of their regions of definition* (8) and *Double distributions and the Dirichlet problem* (9), which appeared in the 1908 Transactions. The first of these takes up the interrelation of the moment $\phi(s)$ of a double distribution along a simple closed curve C: $x=x(s)$, $y=y(s)$ (s being arc length), and the corresponding potential function $W(x, y)$

* See F. Noether, *Über eine Klasse singulärer Integralgleichungen*, Mathematische Annalen, vol. 82 (1921), p. 46.

for (x, y) within C. Under a "Dini condition" upon the derivative $\phi'(s)$ of $\phi(s)$, and a condition

$$|x'(s + \Delta s) - x'(s)|, \ |y'(s + \Delta s) - y'(s)| < N|\Delta s|^\alpha, \quad (\alpha > 0),$$

holding uniformly for $|\Delta s|$ small, he shows that $W(x, y)$, as defined by

$$W(\xi, \eta) = \int_0^l \phi(t) \frac{\partial}{\partial t} \arctan \frac{y(t) - \eta}{x(t) - \xi} dt,$$

is not only continuous within and on the boundary of C, but (with its definition suitably extended) is continuous together with its first partial derivatives in the *closed* region bounded by C. The second paper lightens considerably the restrictions which Fredholm had imposed in his solution of the problem of Dirichlet for the plane; in fact Kellogg imposes only the requirement stated above (whereas Fredholm requires $x(s)$, $y(s)$ to be continuous together with their first three derivatives), while the boundary values $f(s)$ need not be continuous at a finite number of points, s_i, provided that $|f(s)| < G|s - s_i|^{\beta - 1}$, $(\beta > 0)$, nearby. Kellogg shows also that if $f'(s)$ satisfies a Dini condition, so will the corresponding $\phi'(s)$ of the double distribution. In Kellogg's earlier papers it had been tacitly assumed that if an interior potential function $u(x, y)$ takes on a boundary distribution $f(s)$, continuous together with its first derivative, along a sufficiently regular curve C such as the circle, then $u(x, y)$ necessarily possesses continuous first partial dervatives in the *closed* region bounded by C. This inaccuracy was remedied, and the application of integral equations to the Dirichlet problem in the plane was satisfactorily rounded out by these two papers of Kellogg.

His next paper (12) in potential theory, *Harmonic functions and Green's integral*, is also a contribution of decided importance. Among other things he shows in it that if like conditions to the above, bearing equally on $x^{(r)}(s)$, $y^{(r)}(s)$, and $f^{(r)}(s)$ instead of on the first derivatives, are satisfied for a finitely connected region R bounded by k distinct closed curves $C_1, \cdots, C_k$, then there exists a uniquely determined potential (harmonic) function $u(x, y)$ in R taking the values $f(s)$ on the boundary, and this function has continuous derivatives of the first r orders throughout the closed region R. He shows that a corresponding moment $\phi(s)$ of a double distribution exists (which gives $u(x, y)$, augmented by a potential function constant on each of the curves $C_1, \cdots, C_k$), and that this moment satisfies the same kind of condition as that imposed on $f(s)$. By the aid of these results he established various fundamental properties of the Green's function $G(\xi, \eta; x, y)$ for such a multiply connected region R. Furthermore, Kellogg is thus enabled to deal with the case where the assigned boundary values $f(s)$ are merely summable in the sense of Lebesgue. It may be remarked that his thesis had touched in an interesting way upon the Dirichlet problem for multiply connected regions of the plane.

After this paper Kellogg turned his attention in other directions, to which I shall refer later. Meanwhile the war intervened; he spent a year at New London as one of a corps of scientific advisors to the government; and was called to Harvard University in 1919, where he remained till his death.

Upon coming to Harvard he began to occupy himself again with potential theory. In his note *An example in potential theory* (22) of 1923 he presented a concrete example which showed how Dirichlet's problem in the plane was solvable for a certain boundary set which was perfect, nowhere dense, and of Borel measure 0. Kellogg introduces here the notion of a "sequence solution" attached to continuous boundary values on a general boundary, later proved to exist always by Wiener.* Furthermore, Kellogg proves that the existence of Lebesgue's "barrier functions" is necessary as well as sufficient for the solution of the Dirichlet problem. In a nearly contemporaneous note, *On the classical Dirichlet problem for general domains* (26), he discusses the behavior of the solution near "regular" and "exceptional" points of the boundary, and studies further Wiener's "capacity" of a bounded point set. An excellent general account of the status of the Dirichlet problem is contained in Kellogg's address of 1926, already referred to, while in his paper of the same year, *Les moyennes arithmétiques dans la théorie du potentiel,* he develops a conveniently general form of construction of a solution which in special cases reduces to Poincaré's method of "balayage," Schwarz's alternating process, or the Kellogg construction for a sequence solution; this general form of construction is used in Kellogg's well known book *Foundations of Potential Theory* (Berlin, 1929).

The question of the distribution of "regular" and "exceptional" points of the boundary interested Kellogg a great deal. In his note *Unicité des fonctions harmoniques* (32) of 1928, he showed that, in the case of the logarithmic potential, any bounded set of points with positive capacity contains regular points. He was unable to obtain an analogous result for the three-dimensional case, despite protracted efforts.†

It remains to refer to his paper *On the derivatives of harmonic functions on the boundary,*‡ in which Kellogg establishes results for the derivatives of a three-dimensional potential along a boundary, analogous to those which he had given earlier in the two-dimensional case. There is also an incomplete paper *Converses of Gauss' theorem on the arithmetic mean* (40), which is being edited by Dr. J. J. Gergen, and which will appear in the Transactions. Here Kellogg considers a continuous function which in a given region is the arithmetic mean of its values on some circle (or sphere). He proves that if the function takes on one of its bounds at an interior point, then it takes on this bound in every neighborhood of the boundary, but that it cannot take on both its upper and its lower bounds at interior points.

We now turn briefly to his papers in other fields, of which we shall mention only a few of the more important and interesting.

In the first of three papers§ appearing in the American Journal of Mathematics (1916–1918) he studies sets of real orthogonal functions $\phi_0(x)$, $\phi_1(x), \cdots$, and showed that if the determinants $|\phi_i(x_j)|$ of order n, $(n=1, 2, \cdots)$, are positive, that is, if $\phi_0(x_0)>0$, $\phi_0(x\phi)\phi_1(x_1)-\phi_0(x_1)\phi_1(x_0)>0$ when

* *Certain notions in potential theory,* Journal of Mathematics and Physics, vol. 3 (1924), pp. 24–51.

† See (31), (32), (33), (34), (38). The papers (33) and (34) were written jointly with F. Vasilesco. Within a few weeks, Evans has succeeded in establishing "Kellogg's Lemma" in a note *Applications of Poincaré's sweeping-out process,* to appear in the Proceedings of the National Academy of Sciences.

‡ (39). See also the interesting note (28).

§ (14), (15), (16).

$x_1 \neq x_0$, etc., then these functions possess various separation and oscillation properties of many well known orthogonal sets in analysis; in the second and third papers he gives means of applying these results to orthogonal sets defined by linear integral and differential equations.

In a short note *On a diophantine problem* (18), Kellogg takes up the question as to the maximum value of any of n positive integers, the sum of whose reciprocals is unity, and gives reasons for his conjecture that this maximum is u_n where $u_{k+1}=u_k(u_k+1)$, $u_1=1$. This conjecture has been verified* and has led to further interesting work.

The joint paper by Kellogg and myself, *Invariant points in function space* (19), resulted from our interest in simple general forms of existence theorems in analysis. The program proposed was to regard such theorems as the extension to "function space" of the theorem that if a segment (a, b) of a line is carried into part of itself (a', b') by a continuous transformation in such wise that (a', b') lies within (a, b) then at least one point is invariant. Kellogg later (35) generalized a lemma of our paper to essentially the following form: If in the equations $F_i(z_1, \cdots, z_m)=c_i$, $(i=1, \cdots, n)$, the functions F_i are analytic in a region R of complex n-dimensional space, then the analytic manifold thus defined has no singular points in R for general values of $c_1, \cdots, c_n$. This result was one which he found serviceable in potential theory.

In a note *On the existence and closure of sets of characteristic functions,*† he showed how the existence and closure of such sets could often be inferred by a method simpler than the closely related classical method of E. Schmidt.

Kellogg was always interested in questions of mechanics. In his paper *Some properties of spherical curves with applications to the gyroscope,*‡ Kellogg establishes some of the more inaccessible properties of gyroscopic motion by means of the simple intrinsic geometric methods introduced by Osgood.

His paper of 1927, *On bounded polynomials in several variables* (29), contains interesting generalizations of results due to Markoff, Bôcher, and S. Bernstein.

At the time of his death Kellogg was in full scientific activity and was generally recognized as one of the foremost leaders in potential theory, a field to which other American mathematicians—in particular, Bôcher, Evans, and Wiener—have also made contributions of the first order of importance. For several years Kellogg had been planning an advanced companion volume to his *Foundations of Potential Theory* and this project was definitely under way. The first volume had met a longfelt want for a rigorous modern treatment of the subject. Indeed it contained a good deal not to be found elsewhere, in particular a proof of Green's divergence theorem under broad conditions upon the region involved and a thorough-going development of existence theorems on the basis of integral equations. If he had been spared to complete the second volume, it would surely have set a new high-water mark in the field.

As I have said, Kellogg's mind was highly original and versatile. This quality was striking on the social side: If a graceful poem were to be written,

---

* D. R. Curtiss, *On Kellogg's diophantine problem,* American Mathematical Monthly, vol. 29 (1922), pp. 380–387.

† (20). See also the note in this Bulletin, (17).

‡ (23). See also (11), (24).

or an amusing monologue were desired—one could ask Kellogg; when he was writing an intimate family letter, he would incorporate a charming sketch in order to round out a description. Above all, he was the first to think of and carry out a thoughtful and generous act.

On the scientific side he showed the same qualities, which led him to many mathematical results. He loved his chosen science and worked consistently and effectively at difficult and important problems, although his scientific modesty was such that he generally underestimated his own work in comparison to that of others.

In the passing of Oliver Dimon Kellogg, American mathematics has lost a distinguished and beloved figure.

List of Publications of Oliver Dimon Kellogg*

(1) *Zur Theorie der Integralgleichung* $A(s,t)-\mathrm{A}(s,t)=\mu\int_0^1 A(s,r)\mathrm{A}(r,t)dr$, Göttinger Nachrichten, 1902, Heft 2, pp. 165–175.

(2) *Zur Theorie der Integralgleichungen und des Dirichletschen Prinzips*, Dissertation, Göttingen, 1902, iv+43 pp.

(3) *Note on Cauchy's integral*, this Bulletin, vol. 10 (1904), pp. 255–257.

(4) *Unstetigkeiten in den linearen Integralgleichungen*, Mathematische Annalen, vol. 58 (1904), pp. 441–456.

(5) *Unstetigkeiten bei den linearen Integralgleichungen, mit Anwendung auf ein Problem von Riemann*, Mathematische Annalen, vol. 60 (1905), pp. 424–433.

(6) *Note on conjugate potentials*, this Bulletin, vol. 13 (1907), pp. 168–170.

(7) *A necessary condition that all the roots of an algebraic equation be real*, Annals of Mathematics, vol. 9 (1908), pp. 97–98.

(8) *Potential functions on the boundary of their regions of definition*, Transactions of this Society, vol. 9 (1908), pp. 39–50.

(9) *Double distributions and the Dirichlet problem*, Transactions of this Society, vol. 9 (1908), pp. 51–66.

(10) *Two theorems in the geometry of continuously turning curves*, American Mathematical Monthly, vol. 15 (1908), pp. 100–105.

(11) *Applications of the Calculus to Mechanics*, (with E. R. Hedrick), Ginn and Co., 1909, vi+113 pp.

(12) *Harmonic functions and Green's integral*, Transactions of this Society, vol. 13 (1912), pp. 109–132.

(13) *Nomograms with points in alignment*, Zeitschrift für Mathematik und Physik, vol. 63 (1914), pp. 159–173.

(14) *The oscillation of functions of an orthogonal set*, American Journal, vol. 38 (1916), pp. 1–5.

(15) *Orthogonal function sets arising from integral equations*, American Journal, vol. 40 (1918), pp. 145–154.

(16) *Interpolation properties of orthogonal sets of solutions of differential equations*, American Journal, vol. 40 (1918), pp. 225–234.

(17) *Note on closure of orthogonal sets*, this Bulletin, vol. 27 (1920), pp. 165–169.

* Short abstracts, book reviews, and his vice-presidential address *A decade of American mathematics*, Science, vol. 53 (1921), pp. 541–548, are omitted.

(18) *On a diophantine problem*, American Mathematical Monthly, vol. 28 (1921), pp. 300–303.

(19) *Invariant points in function space*, (with G. D. Birkhoff), Transactions of this Society, vol. 23 (1922), pp. 96–115.

(20) *On the existence and closure of sets of characteristic functions*, Mathematische Annalen, vol. 86 (1922), pp. 14–17.

(21) *A property of certain functions whose Sturmian developments do not terminate*, this Bulletin, vol. 28 (1922), pp. 388–389.

(22) *An example in potential theory*, Proceedings of the American Academy of Arts and Sciences, vol. 58, No. 14 (1923), pp. 527–533.

(23) *Some properties of spherical curves with applications to the gyroscope*, Transactions of this Society, vol. 25 (1923), pp. 501–524.

(24) *The theorem on the moment of momentum*, American Mathematical Monthly, vol. 31 (1924), pp. 429–432.

(25) *Proposal and solution of problem 3043*, American Mathematical Monthly, vol. 31 (1924), pp. 499–500.

(26) *On the classical Dirichlet problem for general domains*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 397–406.

(27) *Recent progress with the Dirichlet problem*, this Bulletin, vol. 32 (1926), pp. 601–625.

(28) *On some theorems of Bôcher concerning isolated singular points of harmonic functions*, this Bulletin, vol. 32 (1926), pp. 664–668.

(29) *On bounded polynomials in several variables*, Mathematische Zeitschrift, vol. 27 (1927), pp. 55–64.

(30) *Les moyens arithmétiques dans la théorie du potentiel*, L'Enseignement Mathématique, vol. 27 (1928), pp. 14–26.

(31) *Some notes on the notion of capacity in potential theory*, Acta Universitatis Francisco-Josephinae, Szeged, vol. 4 (1928), pp. 1–5.

(32) *Unicité des fonctions harmoniques*, Comptes Rendus, vol. 187 (1928), pp. 526–527.

(33) *Contribution à l'étude de la capacité et de la série de Wiener*, (with Dr. F. Vasilesco), Comptes Rendus, vol. 188, I, pp. 135–137.

(34) *A contribution to the theory of capacity*, (with Dr. F. Vasilesco), American Journal, vol. 51 (1929), pp. 515–526.

(35) *Singular manifolds among those of an analytic family*, this Bulletin, vol. 35 (1929), pp. 711–716.

(36) *Foundations of Potential Theory*, Berlin, 1929, Julius Springer, vi +384 pp.

(37) *Two notes on instruction in mechanics*, American Mathematical Monthly, vol. 27 (1930), pp. 521–524.

(38) *On the capacity of sets of Cantor type*, American Journal, vol. 53 (1931), pp. 435–482.

(39) *On the derivatives of harmonic functions on the boundary*, Transactions of this Society, vol. 33 (1931), pp. 486–510.

(40) *Converses of Gauss' theorem on the arithmetic mean*, to be published in the Transactions of this Society, as edited by Dr. J. J. Gergen.

G. D. Birkhoff