BATEMAN ON MATHEMATICAL PHYSICS

Partial Differential Equations of Mathematical Physics. By H. Bateman. Cambridge, University Press; NewYork, The Macmillan Co., 1932. 522+xii pp.

The primary purpose of this book is the solution of the boundary-value problems of mathematical physics by means of definite analytical expressions, and the book is noteworthy in fulfilling this object. Some particular topics, such as integral equations, existence theorems of potential theory, Sturm-Liouville series, have been intentionally omitted. Nevertheless, a large number of topics are here treated, and in a unified way. Let us look over the table of contents.

The Introduction discusses briefly the relation of the differential equations to variational principles, and approximate solution of boundary-value problems, method of Ritz, orthogonal functions.

Chapter 1 is on the classical equations, and includes uniform motion, Fourier series, free and forced vibrations, Heaviside's expansion, wave motion, potentials, Laplace's equation, characteristics.

Chapter 2 consists primarily of applications of the theorems of Green and Stokes: Riemann's method, adjoint equations, difference equations as approximations to differential equations, variational principles, transformation of equations, elastic solids, fluid motion, torsion, membranes, electromagnetism, and so on.

Chapter 3 is on two-dimensional problems, Fourier inversion, vibration of a loaded string and of a shaft, Poisson's integral, logarithmic potentials with applications.

Chapter 4 is primarily theoretical, on conformal mapping, including the Riemann theorem, the distortion theorem, Green's function, mapping of polygons, orthogonal polynomials, approximation to the mapping function.

Chapter 5 deals with equations in three variables, wave motion, heat flow. Chapter 6 is on polar coordinates, Legendre polynomials, with generalizations and applications.

Chapter 7 discusses cylindrical coordinates, diffusion, vibration of a circular membrane, Bessel's functions, etc. Chapters 8 and 9 include elliptic and parabolic coordinates, with the corresponding boundary-value problems, including the study of Sonine's polynomials.

Chapter 10 is on toroidal coordinates and applications, Chapter 11 on diffraction problems, and Chapter 12 on non-linear equations, with particular reference to Riccati's equation and applications, minimal surfaces, and the motion of a compressible fluid.

It is clear from this incomplete list that an enormous amount of material is here gathered together and made available. Much of it comes from current (even the most recent) periodical literature, and includes results found by both theory and experiment. The magnitude of the undertaking may be appreciated from the fact that the author index contains 1198 references. The material as a whole is well selected, digested, and correlated. Many results which for one reason or another do not find place in the text are presented as problems to be solved by the reader; this is a most welcome feature. The exposition in general is clear and adequate. Notable is the inclusion of considerable material on aerodynamics.

One might well be content with the work of exposition alone, but in addition there are a number of new results from the author. These are, in particular, new solutions of differential equations (as one might expect from the author's pen), relations between specific solutions of definite equations, and applications. The paragraph on the geometry of lines of force and equipotentials is especially well worked out. The formal solution of a differential equation which is elliptic in one region and hyperbolic in another is of much interest, and deserves further study.

It seems ungrateful to point out faults in such a valuable work, but there are a few relatively unimportant remarks that are not out of place in a review of the book.

The work is of value both as a text and as a book of reference. It is, however, not clear to the reader and seems not always to have been clear to the writer just what sort of book he was writing—an elementary text, an intermediate text, a reference work, or a mathematical diary. Let us take by way of illustration the treatment of Fourier's series, pages 7–16. The introduction is made by means of the equation

$$(d^2y/dx^2)+f(x)=0$$

"and a prescribed set of supplementary conditions," of which no further details are given. The naturalness of Fourier's series as here used is not likely to appeal to any beginner in the subject, for the differential equation itself can be integrated immediately. Why complicate matters by introducing an infinite series? Yet a natural introduction to the topic of Fourier's series is readily given, and indeed an admirable heuristic treatment was long ago given by Byerly.* The discussion given by Bateman is also not suited as an exposition for the more advanced, for the actual convergence of Fourier's series is not proved except for periodic functions which are continuous and with piecewise continuous derivatives. In spite of this criticism of Bateman's treatment, it must be recognized that the treatment (including Parseval's theorem) is interestingly put together, and has elements of novelty and originality.

There are a number of minor defects in the text, mostly due to oversight, or to lack of rigor of the mathematical analysis, of which we mention a few examples. In mathematical rigor, however, the book stands high among other treatments where the primary emphasis is physical.

On page 9 it is supposed that the function $f(x)$ is of bounded variation and *in addition* that the Riemann integrals for its Fourier coefficients exist; the latter condition naturally follows from the former. On page 11, equation (4) would seem to need modification. In the study of the bilinear form on page 37 it is essential to require $g_{rs}=g_{sr}$. On page 153 in the use of a variational principle to prove the existence of a solution of the problem of Dirichlet, reference might suitably be made to the work of Hilbert† and of Lebesgue.‡ On

* *Fourier's Series and Spherical Harmonics*, Boston, 1893, pp. 3 ff.

† Jahresbericht der deutschen Mathematiker Vereinigung, vol. 8 (1900), p. 184.

‡ Palermo Rendiconti, vol. 24 (1907), p. 371.

page 266, in connection with the transformation by an analytic function of a complex variable $\zeta = F(z)$, we read the equation

$$d\zeta = dz \cdot F'(z), \text{ approximately.}$$

It would be more in accord with modern notation to define $d\zeta$ and dz as differentials instead of increments and to omit the word *approximately*. In the study of the bilinear transformation $\zeta = (az+b)/(\alpha z+\beta)$ on page 270 it is essential to require the non-vanishing of the determinant $a\beta - b\alpha$. It would be an aid to the reader if (page 276) the term *region* were defined, for use in the study of the conformal map. To be sure, the term *area* is defined (page 278), but this would seem not to be the term *region* as used on page 291. On page 291 in the proof of the theorem on conformal mapping, it would aid in clearness if the theorem that the uniform limit of a sequence of smooth (i.e. schlicht) analytic functions is smooth were stated explicitly if not proved. On page 325, reference might properly be made to the orthogonal polynomials considered by Carleman.* On page 328 we read "the co-ordinates of a point on a simple closed Jordan curve can be expressed as Fourier series with θ as parameter. The theorem implies that the series (1) converges uniformly throughout d and that the mapping by means of the function $f(z)$ may be extended to regions which are slightly larger than d and D." The latter part of the second sentence might well be omitted.

As a general criticism, one might wish that there were a few more figures—for instance in the study of Riemann's method on page 127—and that the main results were habitually italicised. This latter would increase the value of the book both as an exposition and for reference.

The actual derivation from physical assumptions of partial differential equations as used in the solution of physical problems would be of service to many a reader. All too few carefully worked out examples are to be found in any standard work.

The critical remarks we have made are clearly of secondary interest. The main feature is that such an enormous amount of material has been well unified and clearly set forth. The book must be in the hands of everyone who is interested in the boundary value problems of mathematical physics. One is not inclined to quarrel with the author for his omission of the theory of integral equations from the book. In fact, this opens to the author the opportunity to write a companion volume on the integral equations of mathematical physics. He does not mention any plans on the subject, but such a volume would be another welcome addition to the literature.

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* Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1922–23).